Chapter 4 :Approximation of the confidence interval

In this section, we discuss how to construct an approximate $(1-\alpha)100\%$ confidence interval for a population parameter θ using its maximum likelihood estimator $\widehat{\theta}$.

The confidence interval is obtained by the following steps:
(1)$\ell(X,\theta)=\prod_{i=1}^{n} f(x_i,\theta)$.

(2)$L(X;\theta)=\log(\ell(X,\theta))$.

(3)if $\frac{\partial^2 L(X;\theta)}{\partial\theta^2}<0$, then $\widehat{\theta}_{MLE}$ is the solution of this equation:

$$\frac{\partial L(X;\theta)}{\partial\theta}=0$$

(4)The Fisher information is $I_n=\mathbf{E}(-\frac{\partial^2 L(X;\theta)}{\partial\theta^2})$.

Then the confidence interval of θ is given by:

$$(\hat{\theta}_{MLE}\pm\frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}})$$

Example 1 : Let X be Gamma random variable with distribution :

$$f(x,\theta)=\frac{\theta^r}{\Gamma(r)}x^{r-1}e^{-\theta x};x\geq 0$$

Let $X=(X_1,.,X_n)$ be n copies of X . Find an approximation C.I $(1-\alpha)100$ % of θ.

Solution 1 :
First,find the maximum likelihood estimator of θ :

$$\begin{aligned}\ell(X,\theta) &= \prod_{i=1}^{n} f(x_i,\theta)\\ &= (\frac{\theta^r}{\Gamma(r)})^n \quad (\prod_{i=1}^{n} x_i)^{r-1} \quad e^{-\theta\sum_{i=1}^{n} x_i}\end{aligned}$$

Then

$$\begin{aligned}L(X;\theta) &= \log(\ell(X,\theta))\\ &= nr\log(\theta)-n\log(\Gamma(r))+(r-1)\sum_{i=1}^{n}\log(x_i)-\theta\sum_{i=1}^{n} x_i\end{aligned}$$

The first derivative of the logarithm of likelihood function is:

$$\frac{\partial L(X;\theta)}{\partial\theta}=\frac{nr}{\theta}-\sum_{i=1}^{n} x_i$$

Setting this derivative to zero and solving for θ :

$$\frac{\partial L(X;\theta)}{\partial\theta}=0$$

Hence, the maximum likelihood estimator of θ is given by:

$$\hat{\theta}=\frac{r}{\bar{x}}$$

The second derivative of the logarithm of the likelihood function is given by:

$$\frac{\partial^2 L(X;\theta)}{\partial\theta^2} = \frac{-nr}{\theta^2}$$

The fisher information of θ is given by:

$$\begin{aligned} I_n &= \mathbf{E}(-\frac{\partial^2 L(X;\theta)}{\partial\theta^2}) \\ &= \mathbf{E}(\frac{nr}{\theta^2}) \\ &= \frac{nr}{\theta^2} \end{aligned}$$

we replace the unknown θ by its estimate θ . $(\hat{\theta} = \frac{r}{\bar{x}})$

$$\hat{I}_n = \frac{n\bar{x}^2}{r}$$

The $100(1-\alpha)$ % approximate confidence interval for θ is:

$$\begin{aligned} (\hat{\theta}_{MLE} &\pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}}) \\ (\frac{r}{\bar{x}} &\pm \frac{Z_{1-\frac{\alpha}{2}}}{\bar{x}\sqrt{\frac{n}{r}}}) \end{aligned}$$

Example 2 : Let X be geometric random variable with distribution :

$$f(x,\theta) = \theta \quad (1-\theta)^{x-1}; x = 1,2,3,...$$

let $X = (X_1,.,X_n)$ be n copies of X . Find an approximation $100(1-\alpha)$ % of θ.

Solution 2 :
First ,find the maximum likelihood estimator of θ :

$$\begin{aligned} \ell(X,\theta) &= \prod_{i=1}^{n} f(x_i,\theta) \\ &= (\theta)^n \quad (1-\theta)^{\sum_{i=1}^{n} x_i - n} \end{aligned}$$

Then

$$\begin{aligned} L(X;\theta) &= \log(\ell(X,\theta)) \\ &= n\log(\theta) + (\sum_{i=1}^{n} x_i - n)\log(1-\theta) \end{aligned}$$

The first derivative of the logarithm of likelihood function is:

$$\frac{\partial L(X;\theta)}{\partial\theta} = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i - n}{1-\theta}$$

Setting this derivative to zero and solving for θ :

$$\frac{\partial L(X;\theta)}{\partial\theta} = 0$$

Then

$$\hat{\theta} = \frac{1}{\bar{x}}$$

we need the second derivative of $L(X;\theta)$:

$$\begin{aligned} \frac{\partial^2 L(X;\theta)}{\partial\theta^2} &= \frac{-n}{\theta^2} - \frac{\sum_{i=1}^{n} x_i - n}{(1-\theta)^2} \\ &= -(\frac{n}{\theta^2} + \frac{\sum_{i=1}^{n} x_i - n}{(1-\theta)^2}) \end{aligned}$$

The fisher information of θ is given by:

$$\begin{aligned} I_n &= \mathbf{E}(-\frac{\partial^2 L(X;\theta)}{\partial\theta^2}) \\ &= \mathbf{E}(\frac{n}{\theta^2} + \frac{\sum_{i=1}^n x_i - n}{(1-\theta)^2}) \end{aligned}$$

mean of a geometric distribution is given as follows: $\mathbf{E}(x) = \frac{1}{\theta}$

$$\begin{aligned} I_n &= n[\frac{1}{\theta^2} - \frac{1}{(1-\theta)^2} + \frac{1}{\theta(1-\theta)^2}] \\ &= n[\frac{(1-\theta)^2 - \theta^2 + \theta}{\theta^2(1-\theta)^2}] \\ &= n[\frac{1-\theta}{\theta^2(1-\theta)^2}] \\ &= \frac{n}{\theta^2(1-\theta)} \end{aligned}$$

we replace the unknown θ by its estimate $\widehat{\theta}$

$$\widehat{I}_n = \frac{n}{\widehat{\theta}^2(1-\widehat{\theta})}$$

Thus , $100(1-\alpha)$ % approximate confidence interval for θ is:

$$\begin{aligned} &(\hat{\theta}_{MLE} \pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}}) \\ &(\hat{\theta} \pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\theta}^2(1-\hat{\theta})}{n}} \quad) \end{aligned}$$

Example 3 : If $X_1, X_2, ..., X_n$ is a random sample from a population with density:

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1. \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta > 0$ is an unknown parameter.
(a) what is a $100(1-\alpha)$ % approximate confidence interval for θ if the sample size is large?
(b) Find 90% approximate C.I of θ , if $\sum_{i=1}^{49} \log x_i = -0.7564$
Solution 3:
(a) First,find the maximum likelihood estimator of θ :

$$\begin{aligned} \ell(X,\theta) &= \prod_{i=1}^n f(x_i,\theta) \\ &= (\theta)^n \prod_{i=1}^n x_i^{(\theta-1)} \end{aligned}$$

Then

$$\begin{aligned} L(X;\theta) &= \log(\ell(X,\theta)) \\ &= n\log(\theta) + (\theta-1)\sum_{i=1}^n \log(x_i) \end{aligned}$$

The first derivative of the logarithm of likelihood function is:

$$\frac{\partial L(X;\theta)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^n \log(x_i)$$

Setting this derivative to zero and solving for θ :

$$\frac{\partial L(X;\theta)}{\partial \theta} = 0$$

Hence, the maximum likelihood estimator of θ is given by:

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \log(x_i)}$$

The second derivative of the logarithm of the likelihood function is given by:

$$\frac{\partial^2 L(X;\theta)}{\partial \theta^2} = \frac{-n}{\theta^2}$$

The fisher information of θ is given by:

$$\begin{aligned} I_n &= \mathbf{E}(-\frac{\partial^2 L(X;\theta)}{\partial \theta^2}) \\ &= \mathbf{E}(\frac{n}{\theta^2}) \\ &= \frac{n}{\theta^2} \end{aligned}$$

we replace the unknown θ by its estimate θ . $(\hat{\theta} = \frac{-n}{\sum \log x_i})$

$$\hat{I}_n = \frac{(\sum_{i=1}^{n} \log x_i)^2}{n}$$

The $100(1-\alpha)$ % approximate confidence interval for θ is:

$$\begin{aligned} (\hat{\theta}_{MLE} &\pm \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}}) \\ (\frac{-n}{\sum \log x_i} &\pm \frac{Z_{1-\frac{\alpha}{2}}\sqrt{n}}{\sum \log x_i}) \end{aligned}$$

(b) We are given the followings:

$$\begin{aligned} n &= 49 \\ \sum_{i=1}^{49} \log x_i &= -0.7576 \\ 1-\alpha &= 0.90 \end{aligned}$$

Hence, we get

$$Z_{0.95} = 1.64$$

$$\frac{n}{\sum \log x_i} = \frac{49}{-0.7567} = -64.75$$

and

$$\frac{\sqrt{n}}{\sum \log x_i} = \frac{7}{-0.7567} = -9.25$$

Hence, the approximate confidence interval is given by

$$(64.75 - (1.64)(9.25), 64.75 + (1.64)(9.25))$$

that is $(49.58, 79.92)$.

Example 4 : class activity

If $X_1, X_2, ..., X_n$ is a random sample from a population with density:

$$f(x;\theta)=\begin{cases}(1-\theta)\theta^x & \text{if } x=0,1,2,.... \\ 0 & \text{otherwise.}\end{cases}$$

where $0<\theta<1$ is an unknown parameter, what is a $100(1-\alpha)$ % approximate confidence interval for θ if the sample size is large?

Solution 4:
Thus $100(1-\alpha)$ % approximate confidence interval for θ is:

$$(\hat{\theta}_{MLE} \quad \pm \quad \frac{Z_{1-\frac{\alpha}{2}}}{\sqrt{\hat{I}_n}})$$
$$(\hat{\theta} \quad \pm \quad Z_{1-\frac{\alpha}{2}}\frac{\hat{\theta}(1-\hat{\theta})}{\sqrt{n(1-\widehat{\theta}+\widehat{\theta}^2)}})$$

where

$$\hat{\theta}=\frac{\bar{x}}{1+\bar{x}}$$

Example 5 : Homework
Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with a probability density function:

$$f(x;\theta)=\begin{cases}(\theta+1)x^{-\theta-2} & \text{if } 1<x<\infty. \\ 0 & \text{otherwise.}\end{cases}$$

where $0<\theta$ is a parameter, what is a $100(1-\alpha)$ % approximate confidence interval for θ if the sample size is large?

Solution 5:

$$\hat{\theta}\pm Z_{1-\frac{\alpha}{2}}\frac{\hat{\theta}+1}{\sqrt{n}}$$

where $\hat{\theta}=-1+\frac{n}{\sum_{i=1}^{n}\log x_i}$