Problem 1: Let X be a random variable with distribution

$$f(x;\theta) = \frac{1}{\theta}\ ;\ \ 0 < x < \theta$$

Let $\underline{X} = (X_1, X_2, \dots, X_n)$ be n copies of X

Find $(1-\alpha)100\%$ Confidence interval for θ. (Hint: use $Q = \frac{S}{\theta}$ where $S = \max(\underline{X})$ and $h(q) = nq^{n-1}\quad ;\ \ 0 < q < 1$).

Solution

STEP 1:

$$P(q_1 < Q(X,\theta) < q_2) = \int_{q_1}^{q_2} h(q)dq = 1-\alpha$$

$$\int_{q_1}^{q_2} h(q)dq = \int_{q_1}^{q_2} n\, q^{n-1}\, dq = n\left[\frac{q^n}{n}\right]_{q_1}^{q_2} = q_2^n - q_1^n = 1-\alpha \qquad (*)$$

$$P\left(q_1 < \frac{S}{\theta} < q_2\right) = P\left(\frac{S}{q_2} < \theta < \frac{S}{q_1}\right) = 1-\alpha$$

STEP 2:

$$L = S\left(\frac{1}{q_1} - \frac{1}{q_2}\right) \quad \textit{must be minimum.} \quad (**)$$

To minimize L you should choose q_1 to be as large as possible (close to 1) and q_2 to be just slightly larger than q_1 because $q_1 < q_2$. So differentiate (*) and (**) with respect q_2

$$\frac{d}{dq_2}\left[\int_{q_1}^{q_2} h(q)dq\right] = 0$$

$$\frac{d}{dq_2}(q_2^n - q_1^n) = 0$$

$$nq_2^{n-1} - nq_1^{n-1}\frac{dq_1}{dq_2} = 0$$

$$\gg\gg \quad \frac{dq_1}{dq_2} = \frac{q_2^{n-1}}{q_1^{n-1}}$$

$$\frac{d}{dq_2}L = S\left(-\frac{1}{q_1^2}\frac{dq_1}{dq_2} + \frac{1}{q_2^2}\right) = S\left(-\frac{1}{q_1^2}\frac{q_2^{n-1}}{q_1^{n-1}} + \frac{1}{q_2^2}\right) = S\left(\frac{1}{q_2^2} - \frac{q_2^{n-1}}{q_1^{n+1}}\right)$$

$$\frac{d}{dq_2}L = \frac{q_1^{n+1} - q_2^{n-1}\, q_2^2}{q_2^2 q_1^{n+1}} = \frac{q_1^{n+1} - q_2^{n+1}}{q_2^2 q_1^{n+1}} < 0$$

$$\frac{d}{dq_2}L < 0 \; because \; q_1 < q_2$$

i.e. L is decreasing function of q_2 and we get the minimum value when $q_2 = 1$ (maximum)

$$P\left(q_1 < \frac{S}{\theta} < q_2\right) = \int_{q_1}^{q_2} n\, q^{n-1}\, dq = q_2^n - q_1^n = 1 - \alpha$$

Substitute $q_2 = 1$ we get

$$1 - q_1^n = 1 - \alpha \ggg q_1^n = 1 - (1 - \alpha) \ggg \; q_1^n = \alpha \ggg \quad q_1 = \alpha^{\frac{1}{n}}$$

$Then \; \frac{S}{q_1} = \frac{S}{\alpha^{\frac{1}{n}}} \; and \; \frac{S}{q_2} = S$

The confidence interval is $\left(S\,, \frac{S}{\alpha^{\frac{1}{n}}}\right)$

Problem 2: Let X be a random variable with distribution

$$f(x;\theta) = e^{-(x-\theta)}\,; \;\; x > \theta$$

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be n copies of X

Find $(1-\alpha)100\%$ Confidence interval for θ. (Hint: use $Q = n(S - \theta)$ where $S = \min(\underline{X})$ and $h(q) = e^{-q}; \quad q > 0$).

Solution

STEP 1:

$$P(q_1 < Q(X, \theta) < q_2) = \int_{q_1}^{q_2} h(q) dq = 1 - \alpha$$

$$\int_{q_1}^{q_2} h(q) dq = -\int_{q_1}^{q_2} (-) e^{-q}\, dq = -[e^{-q}]_{q_1}^{q_2} = -e^{-q_2} + e^{-q_1}$$

$$= \boldsymbol{e^{-q_1} - e^{-q_2}} = 1 - \alpha \quad (*)$$

$$P(q_1 < n(S - \theta) < q_2) = P\left(\frac{q_1}{n} < S - \theta < \frac{q_2}{n}\right) = P\left(\frac{q_1}{n} - S < -\theta < \frac{q_2}{n} - S\right) = 1 - \alpha$$

$$P\left(S - \frac{q_2}{n} < \theta < S - \frac{q_1}{n}\right) = 1 - \alpha$$

STEP 2:

$$L = S - \frac{q_1}{n} - \left(S - \frac{q_2}{n}\right) = \frac{1}{n}(q_2 - q_1) \qquad \textit{must be minimum.} \quad (**)$$

To minimize L you should choose q_1 to be as large as possible. So differentiate (*) and (**) with respect q_1.

$$\frac{d}{dq_1}\left[e^{-q_1} - e^{-q_2}\right] = 0 \ \ggg\ -e^{-q_1} - (-e^{-q_2})\,\frac{dq_2}{dq_1} = 0$$

$$\frac{dq_2}{dq_1} = \frac{e^{-q_1}}{e^{-q_2}} = \boldsymbol{e^{-q_1+q_2}}$$

$$\frac{d}{dq_1}L = \frac{1}{n}\left(\frac{dq_2}{dq_1} - 1\right) = \frac{1}{n}(\boldsymbol{e}^{\,q_2-q_1} - 1\,) > 0$$

Note:

n is just a constant $n > 0$ and $q_2 - q_1 > 0$ because $q_2 > q_1$, therefore $\boldsymbol{e}^{\,q_2-q_1} > \mathbf{1.}$

If $\frac{dL}{dq_1} > 0$, then L increases as q_1 increases.

If $\frac{dL}{dq_1} < 0$, then L increases as q_1 decreases.

So, L is increasing function of q_1 and we get the minimum value when $q_1 = 0$ (minimum value)

$$\int_{q_1}^{q_2} h(q)dq = \boldsymbol{e^{-q_1}} - \boldsymbol{e^{-q_2}} = 1 - \alpha$$

Substitute $q_1 = 0$ we get

$$\mathbf{1} - \boldsymbol{e^{-q_2}} = 1 - \alpha \ggg \quad \boldsymbol{e^{-q_2}} = \alpha \ggg -\mathrm{q}_2 = \ln\alpha \ggg \ \mathbf{q_2} = \boldsymbol{-\ln\alpha}$$

$$S - \frac{q_1}{n} = S \quad ; \quad S - \frac{q_2}{n} = S + \frac{\ln\alpha}{n}$$

The confidence interval is $\left(S + \frac{\ln\alpha}{n}, S\right)$.