

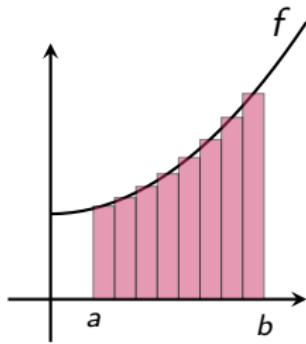
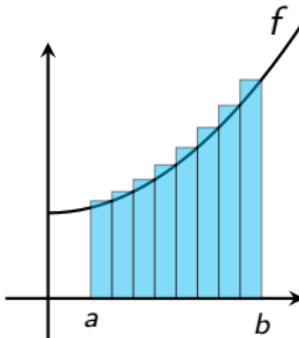
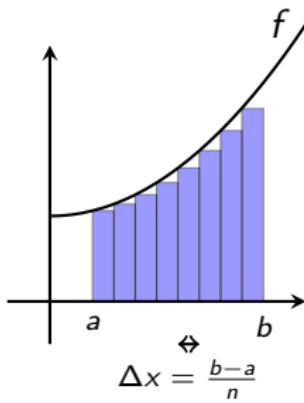
# Multiple Integrals

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In calculus of one variable, we use the theory of Riemann integration to compute the area under the graph of a function  $f : [a, b] \rightarrow \mathbb{R}$ . This integral is approximated by areas of a collection of rectangles.



## Definition

Consider a region  $\Omega$  of  $\mathbb{R}^2$ . Assume that  $\Omega$  can be divided into a finite number of horizontal and vertical lines. Then the totality of closed rectangles that lie completely within  $\Omega$  is called an inner partition of  $\Omega$ .

## Definition

Let  $f: \Omega \rightarrow \mathbb{R}$  be a function of two variables defined on a region  $\Omega$  and let  $P = (R_k)_{1 \leq k \leq n}$  be an inner partition of  $\Omega$ . For any mark  $(x_k, y_k) \in P_k$ , consider the Riemann sum

$$R(f, \Omega, P) = \sum_{k=1}^n f(x_k, y_k) A_k,$$

where  $A_k$  is the area of the rectangle  $P_k$ .

## Theorem

Let  $f: \Omega \rightarrow \mathbb{R}$  be a function of two variables defined on a region  $\Omega$ . The double integral of  $f$  over  $\Omega$ , denoted by  $\iint_{\Omega} f(x, y) dx dy$ ,

is  $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) A_k$ , provided the limit exists.

If the double integral of  $f$  over  $\Omega$  exists, then  $f$  is said to be integrable over  $\Omega$ . It can be proved that if  $f$  is continuous on  $\Omega$ , then  $f$  is integrable over  $\Omega$ .

# Interpretation of the Double Integral

A useful geometric interpretation for double integral for a non negative continuous function  $f$  throughout a region  $\Omega$ . Let  $S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$  the surface defined by  $f$  and let  $V$  be the solid that lies under  $S$  and over  $\Omega$ . The volume of  $V$  is

$$\iint_{\Omega} f(x, y) dx dy.$$

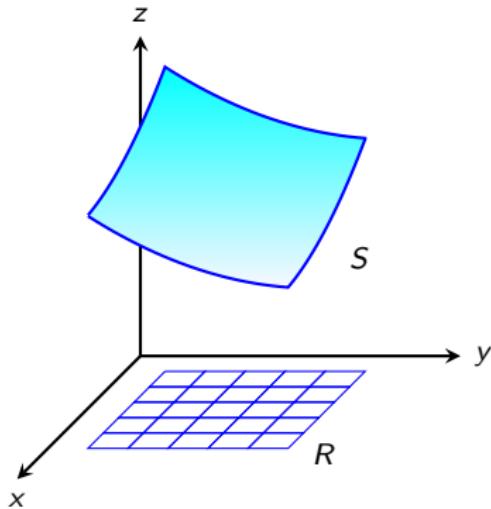
To find the volume between the rectangle  $R = [a, b] \times [c, d]$  in the  $xy$  plane and the surface

$$S = \{(x, y, z) : (x, y) \in R, z = f(x, y)\}$$

where  $f$  is a continuous function, we proceed as follows: We divide the rectangle  $R$  into an  $n \times n$  subsquares  $R_{i,j}$ . For each subsquare  $R_{i,j}$ , we make a box of height  $f(x_i, y_j)$ , where  $(x_i, y_j) \in R_{i,j}$ .

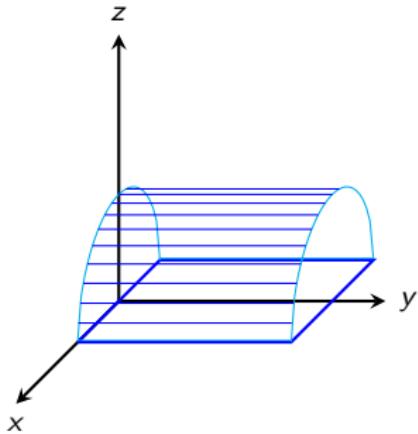
We approximate this volume by adding up the volumes of the  $n^2$  boxes. The exact volume is obtained by taking the limit when  $n \rightarrow +\infty$ .

If  $\Delta_n$  is the area of rectangle  $R_{i,j}$ , the approximate volume under the surface is  $\sum_{i=1}^n \sum_{j=1}^n f(x_i, y_j) \Delta_n$ .



If  $R = [a, b] \times [c, d]$  and  $S = \{(x, y, z) : z = f(x, y), (x, y) \in R\}$ , then the volume between  $R$  and  $S$  is

$$V = \lim_{n \rightarrow \infty} \sum_{i,j=1}^n f(x_i, y_j) \Delta_n = \iint_R f(x, y) dx dy.$$



# Evaluation of Double Integrals

## Definition

A subset  $\Omega$  of  $\mathbb{R}^2$  is called elementary if there exist  $a, b, c, d \in \mathbb{R}$  with  $a < b$  and  $c < d$ , and functions  $\varphi_1, \varphi_2$  continuous on  $[a, b]$  and  $\psi_1, \psi_2$  continuous on  $[c, d]$  such that  $\varphi_1(x) \leq \varphi_2(x)$  for all  $x \in [a, b]$ ,  $\psi_1(y) \leq \psi_2(y)$  for all  $y \in [c, d]$  and

$$\begin{aligned}\Omega &= \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}\end{aligned}$$

# Example

- ① The rectangle  $[a, b] \times [c, d]$ , with  $a < b$  and  $c < d$  is an elementary subset of  $\mathbb{R}^2$ .
- ② The unit disc  $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$  can be written as

$$D = \{(x, y) \in \mathbb{R}^2 | -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$$

and

$$D = \{(x, y) \in \mathbb{R}^2 | -1 \leq y \leq 1, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}.$$

- ③ The ring  $\{(x, y) \in \mathbb{R}^2 | 1 \leq \sqrt{x^2 + y^2} \leq 2\}$  is a simple domain of  $\mathbb{R}^2$ .

# Bubini's Theorem

theorem

[Fubini]

Let  $\Omega$  be an elementary subset of  $\mathbb{R}^2$  and  $f$  a continuous function on  $A$ . With the same notations as preview, we have

$$\int_{\Omega} f(x, y) dx dy = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right) dx = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \right) dy$$

## Remark

The previous theorem can be readied in two different manners. The construction of the integral of a continuous function on a domain  $A$  of  $\mathbb{R}^2$ , the Fubini's theorem says that we can do the integral with respect to  $x$  and  $y$  is constant and then we integrate with respect to  $y$  and vice versa.

We usually denote  $\iint_{\Omega} f(x, y) dx dy$  for this integral.

# Example

Consider the function  $f(x, y) = xy^2$  on the rectangle  $R = [0, 1] \times [1, 2]$ . We have

$$\begin{aligned}\int_R xy^2 dxdy &= \int_0^1 \left( \int_1^2 xy^2 dy \right) dx \\ &= \int_0^1 \left( \frac{8x}{3} - \frac{x}{3} \right) dx = \int_0^1 \frac{7x}{3} dx = \frac{7}{6}.\end{aligned}$$

Also we have

$$\begin{aligned}\int_R xy^2 dxdy &= \int_1^2 \left( \int_0^1 xy^2 dx \right) dy \\ &= \int_1^2 \frac{y^2}{2} dy = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.\end{aligned}$$

# Example

Consider the domain

$$T = \{(x, y) \in [0, 1]^2 : y \leq x\}.$$

$$\begin{aligned}\iint_T x^2 y^3 dx dy &= \int_0^1 \left( \int_0^x x^2 y^3 dy \right) dx \\ &= \int_0^1 \frac{x^6}{4} dx = \frac{1}{28}.\end{aligned}$$

Now using horizontal strips, we get:

$$\begin{aligned}\iint_T x^2 y^3 dx dy &= \int_0^1 \left( \int_y^1 x^2 y^3 dx \right) dy \\ &= \frac{1}{3} \int_0^1 y^3 (1 - y^3) dy = \frac{1}{28}.\end{aligned}$$

# Example

Consider the domain  $\Omega = \{1 \leq x \leq 3, \frac{x}{3} \leq y \leq \frac{4x}{3}\}$ .

$$\begin{aligned}\iint_{\Omega} (x + 3y) dx dy &= \int_1^3 \left( \int_{\frac{x}{3}}^{\frac{4x}{3}} (x + 3y) dy \right) dx \\ &= \frac{7}{2} \int_1^3 x^2 dx = \frac{91}{3}.\end{aligned}$$

# Example

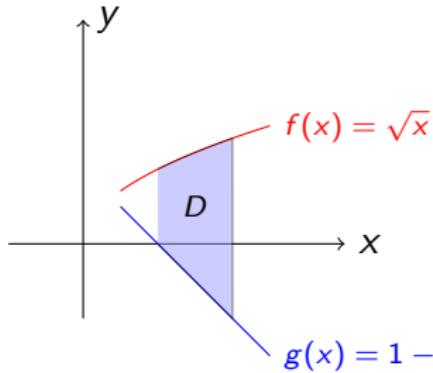
Evaluation the integral  $\int_1^2 \int_{1-x}^{\sqrt{x}} x^2 y dy dx$

$$\begin{aligned}\int_1^2 \int_{1-x}^{\sqrt{x}} x^2 y dy dx &= \int_1^2 \frac{1}{2} x^2 (x - (1-x)^2) dx \\ &= \frac{1}{2} \int_1^2 (3x^3 - x^2 - x^4) dx = 6 - \frac{7}{6} - \frac{3}{2} - \frac{31}{10}.\end{aligned}$$

Consider the domain

$D = \{(x, y) \in \mathbb{R}^2 : 1 - x \leq y \leq \sqrt{x}, 1 \leq x \leq 2\}$ . We have

$$\int_1^2 \int_{1-x}^{\sqrt{x}} x^2 y dy dx = \iint_D x^2 y dxdy.$$



The area of  $D$  is  $A = \int \int_D dxdy = \int_1^2 \int_{1-x}^{\sqrt{x}} dy dx = \frac{4\sqrt{2}}{3} - \frac{1}{6}$ .

Now using horizontal strips, we get:

$$A = \int \int_D dxdy$$

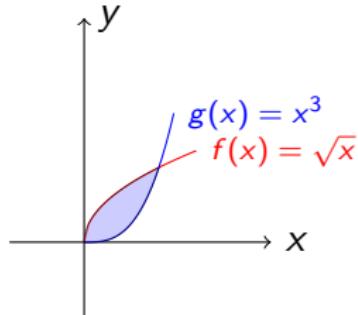
$$\int_0^{\sqrt{2}} \int_{\frac{1}{2}y^2}^2 dxdy$$

# Example

Evaluation the integral  $\int_0^1 \int_{x^3}^{\sqrt{x}} xy dy dx$

$$\int \int_D xy dx dy = \int_0^1 \int_{x^3}^{\sqrt{x}} xy dy dx = \frac{1}{2} \int_0^1 x^2 (1 - x^5) dx = \frac{5}{48},$$

where  $D = \{(x, y) \in \mathbb{R}^2 : x^3 \leq y \leq \sqrt{x}, 0 \leq x \leq 1\}$ .



Now using horizontal strips, we get:

$$\int_0^1 \int_{x^3}^{\sqrt{x}} xy dy dx = \int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} xy dx dy = \frac{1}{2} \int_0^1 y (y^{\frac{2}{3}} - y^4) dy = \frac{5}{48}.$$

## definition

A subset  $\Omega$  of  $\mathbb{R}^2$  is called simple if it is a finite union of elementary subsets  $\Omega_1, \dots, \Omega_n$  with disjoint interiors.

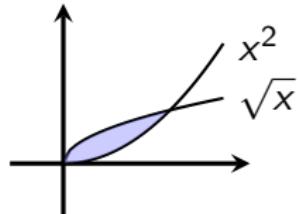
$$\forall 1 \leq i, j \leq n, \quad i \neq j, \quad \Omega_i \cap \Omega_j = \emptyset.$$

If  $f$  is a continuous function on  $\Omega$ , we define

$$\iint_{\Omega} f(x, y) dx dy = \sum_{k=1}^n \iint_{\Omega_k} f(x, y) dx dy.$$

# Example

Consider the region bounded by the graphs of  $y = \sqrt{x}$  and  $y = x^3$



If  $f$  is a continuous function on the region  $R$ ,

$$\iint_R f(x, y) dxdy = \int_0^1 \left( \int_{x^3}^{\sqrt{x}} f(x, y) dy \right) dx$$

and

$$\iint_R f(x, y) dxdy = \int_0^1 \left( \int_{y^2}^{y^{\frac{1}{3}}} f(x, y) dx \right) dy.$$

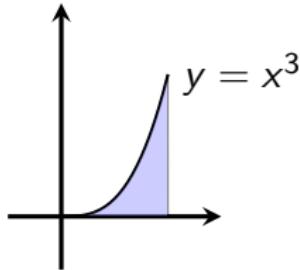
# Example

$$\begin{aligned}\int_1^e \left( \int_0^{\ln x} y dy \right) dx &= \int_1^e \int_0^{\ln x} y dx dy \\&= \int_1^e \frac{1}{2} \ln^2 x dx \\&= \left. \frac{1}{2} x \ln^2 x - x \ln x + x \right]_1^e = \frac{1}{2}(e-2).\end{aligned}$$

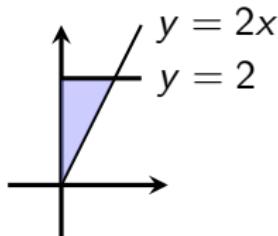
$$\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{y}{\sqrt{16+x^7}} dx dy.$$

Changing to vertical strip,  $0 \leq y \leq x^3$ ,  $0 \leq x \leq 2$ ,

$$\begin{aligned} \int_0^8 \int_{\sqrt[3]{y}}^2 \frac{y}{\sqrt{16+x^7}} dx dy &= \int_0^2 \left( \int_0^{x^3} \frac{y}{\sqrt{16+x^7}} dy \right) dx \\ &= \frac{1}{2} \int_0^2 \frac{x^3}{16+x^7} dx = \frac{8}{7}. \end{aligned}$$



$$\int_0^1 \int_{2x}^2 e^{y^2} dx dy.$$



Changing to horizontal strip,  $0 \leq y \leq 1$ ,  $0 \leq x \leq \frac{y}{2}$ ,

$$\begin{aligned} \int_0^1 \int_{2x}^2 e^{y^2} dx dy &= \int_0^2 \int_0^{\frac{y}{2}} e^{y^2} dx dy \\ &= \frac{1}{2} \int_0^2 y e^{y^2} dy = \frac{1}{4}(e^4 - 1). \end{aligned}$$

# Example

$f(x, y) = x + y$  and  $R = [0, 1] \times [1, 2]$ . Then

$$\begin{aligned}\iint_R f(x, y) dxdy &= \int_1^2 \left( \int_0^1 f(x, y) dx \right) dy \\ &= \int_1^2 \left( \frac{1}{2} + y \right) dy = 2.\end{aligned}$$

Also,

$$\begin{aligned}\iint_R f(x, y) dxdy &= \int_0^1 \left( \int_1^2 f(x, y) dy \right) dx \\ &= \int_0^1 \left( x + \frac{3}{2} \right) dx = 2.\end{aligned}$$

# Example

Let  $f(x, y) = xy$  and  $S$  the region bounded by the curves  $x = y^2$  and  $x = y$   $S = \{(x, y) \in \mathbb{R}^2 : y^2 \leq x \leq y, 0 \leq y \leq 1\}$ .

$$\begin{aligned}\iint_S xy \, dxdy &= \int_0^1 dy \int_{y^2}^y xy \, dx \\ &= \int_0^1 y \left( \int_{y^2}^y x \, dx \right) dy \\ &= \frac{1}{2} \int_0^1 (y^3 - y^5) \, dy = \frac{1}{24}.\end{aligned}$$

In this case we can also represent  $S$  in the form

$S = \{(x, y) \in \mathbb{R}^2 : x \leq y \leq \sqrt{x}, 0 \leq x \leq 1\}$ . Hence,

$$\begin{aligned}\iint_S xy \, dxdy &= \int_0^1 x \, dx \int_x^{\sqrt{x}} y \, dy = \int_0^1 \left( \frac{y^2}{2} \Big|_{y=x}^{\sqrt{x}} \right) x \, dx \\ &= \frac{1}{2} \int_0^1 (x^2 - x^3) \, dx = \frac{1}{24}.\end{aligned}$$

# Example

Let  $f(x, y) = x + y$  and

$$S = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 0 \leq y \leq 1 + |x|\}.$$

$$\iint_S (x, y) dxdy = \iint_{S_1} (x + y) dxdy + \iint_{S_2} (x + y) dxdy, \text{ where}$$

$$S_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1 + x\} \text{ and}$$

$$S_2 = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 0, 0 \leq y \leq 1 - x\}.$$

$$\begin{aligned}
 \iint_{S_1} (x+y) dx dy &= \int_0^1 dx \int_0^{1+x} (x+y) dy \\
 &= \frac{1}{2} \int_0^1 [(2x+1)^2 - x^2] dx \\
 &= \frac{1}{2} \left[ \frac{(2x+1)^3}{6} - \frac{x^3}{3} \right] \Big|_0^1 = 2
 \end{aligned}$$

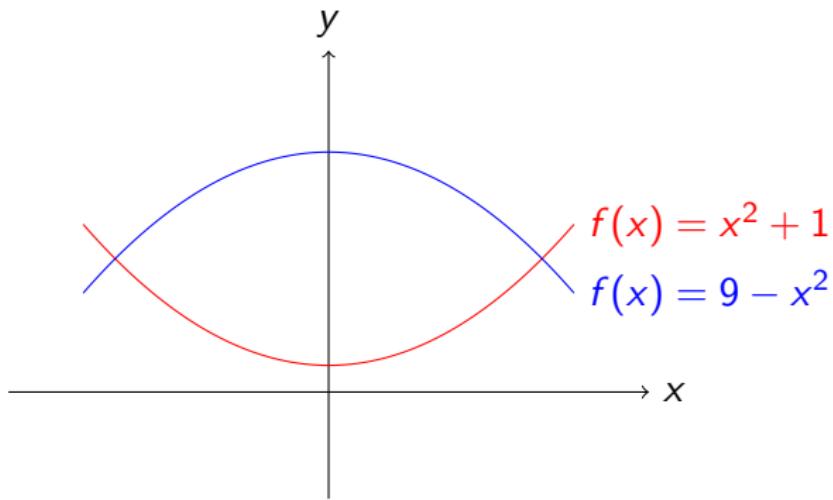
and

$$\begin{aligned}
 \iint_{S_2} (x+y) dx dy &= \int_{-1}^0 dx \int_0^{1-x} (x+y) dy \\
 &= \frac{1}{2} \int_{-1}^0 (1-x^2) dx = \frac{1}{2} \left( x - \frac{x^3}{3} \right) \Big|_{-1}^0 = \frac{1}{3}.
 \end{aligned}$$

$$\text{Therefore, } \iint_S (x+y) dx dy = 2 + \frac{1}{3} = \frac{7}{3}.$$

# Example

Consider the region bounded by the curves  $y = x^2 + 1$  and  $y = 9 - x^2$ .  $S = \{(x, y) \in \mathbb{R}^2 : x^2 + 1 \leq y \leq 9 - x^2, -2 \leq x \leq 2\}$ .



$$\begin{aligned} A = \iint_S dxdy &= \int_{-2}^2 dx \int_{x^2+1}^{9-x^2} dy \\ &= \int_{-2}^2 [(9 - x^2) - (x^2 + 1)] \, dx \\ &= \int_{-2}^2 (8 - 2x^2) \, dx = \frac{64}{3}. \end{aligned}$$

# Exercises

## Exercise 1 :

Compute the following integrals

①  $\iint_R x \sec^2 y dx dy, R = [0, 2] \times [0, \pi/4]$

②  $\iint_R \frac{xy^2}{x^2 + 1} dx dy, R = [0, 1] \times [-3, 3]$

③  $\iint_R \frac{1}{1+x+y} dx dy, R = [1, 3] \times [1, 2]$

**Exercise 2 :**

For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed.  
Evaluate the integrals both ways.

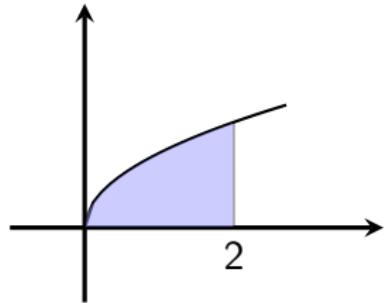
$$\textcircled{1} \quad \int_0^{\sqrt{2}} \int_{y^2}^2 y \, dx \, dy;$$

$$\textcircled{2} \quad \int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, dy \, dx;$$

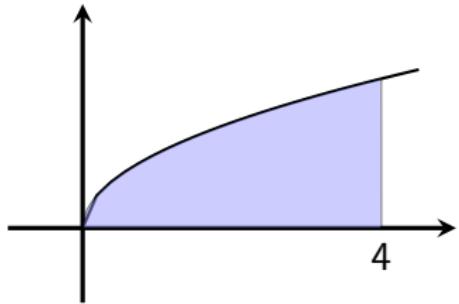
$$\textcircled{3} \quad \int_0^1 \int_{-y}^{y^2} x \, dx \, dy.$$

**Solution**

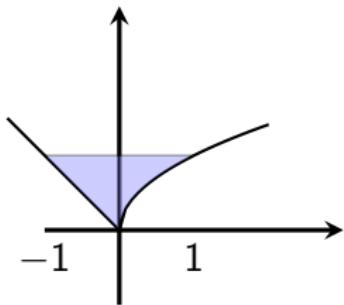
$$\int_0^{\sqrt{2}} \int_{y^2}^2 y \, dx \, dy = \int_0^2 \int_0^{\sqrt{x}} y \, dy \, dx = \frac{1}{2} \int_0^2 x \, dx = 1.$$



$$\int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, dy \, dx = \int_0^2 \int_{y^2}^4 y\sqrt{x} \, dx \, dy = \frac{2}{3} \int_0^2 (8 - y^3) y \, dy = \frac{42}{5}.$$



$$\int_0^1 \int_{-y}^{y^2} x \, dx \, dy = \int_{-1}^0 \int_{-x}^1 dy \, dx + \int_0^1 \int_{\sqrt{x}}^1 dy \, dx = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$



**Exercise 3 :**

Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \int_y^{\pi} x^{-1} \sin x \, dx \, dy .$$

**Solution**

$$\int_0^{\pi} \int_y^{\pi} x^{-1} \sin x \, dx \, dy = \int_0^{\pi} \int_0^x x^{-1} \sin x \, dy \, dx = \int_0^{\pi} \sin x \, dx = 2.$$

**Exercise 4 :**

For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed.  
Evaluate the integrals both ways.

$$\textcircled{1} \quad \int_0^{\sqrt{2}} \int_{y^2}^2 y \, dx \, dy,$$

$$\textcircled{2} \quad \int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, dy \, dx,$$

$$\textcircled{3} \quad \int_0^1 \int_{-y}^{y^2} x \, dx \, dy .$$

①  $\int_0^{\sqrt{2}} \int_{y^2}^2 y \, dx \, dy = \int_0^{\sqrt{2}} y(2 - y^2) \, dy = 1.$  Moreover,

$$\int_0^{\sqrt{2}} \int_{y^2}^2 y \, dx \, dy = \int_0^2 \int_0^{\sqrt{x}} y \, dy \, dx = \int_0^2 \frac{1}{2}x \, dx = 1.$$

②  $\int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, dy \, dx = \frac{1}{2} \int_0^4 x^{\frac{3}{2}} \, dx = \frac{32}{5}.$  Moreover,

$$\int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, dy \, dx = \int_0^2 y \int_{y^2}^4 \sqrt{x} \, dx \, dy =$$

③  $\int_0^1 \int_{-y}^{y^2} x \, dx \, dy = \int_0^1 (y^4 - y^2) \, dy = -\frac{1}{15}.$  Moreover,

$$\int_0^1 \int_{-y}^{y^2} x \, dx \, dy = \int_{-1}^0 x \int_{-x}^1 \, dy \, dx + \int_0^1 x \int_{\sqrt{x}}^1 \, dy \, dx =$$

$$-\frac{1}{6} + \frac{1}{10} = -\frac{1}{15}.$$

**Exercise 5 :**

Reverse the order of integration and hence evaluate:

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} dx dy .$$

$$\begin{aligned}\int_0^\pi \int_y^\pi \frac{\sin x}{x} dx dy &= \int_0^\pi \int_0^x \frac{\sin x}{x} dy dx \\ &= \int_0^\pi \sin x dx = 2.\end{aligned}$$