Use LU-factorization method with Doolittle's method ($l_{ii} = 1$) to find the solution of the consistent system for $\alpha \neq 3$.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & & & = & 1 \\ 3x_1 & + & \alpha x_2 & + & 5x_3 & = & 8 \\ & & 7x_2 & + & 3x_3 & = & 3 \end{array}$$

Solution:

هذا الهامش

We use Simple Gauss-elimination method to convert $A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & \alpha & 5 \\ 0 & 7 & 3 \end{pmatrix}$ into an upper-triangular matrix

$m_{21} = \frac{3}{1} = 3$, $m_{31} = \frac{0}{1} = 0 \Rightarrow \xrightarrow[-m_{31}R_1 + R_3]{-m_{21}R_1 + R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha-3 & 5 \\ 0 & 7 & 3 \end{pmatrix}$

$m_{32} = \frac{7}{\alpha-3} \Rightarrow \xrightarrow{-m_{32}R_2 + R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha-3 & 5 \\ 0 & 0 & 3-\frac{35}{\alpha-3} \end{pmatrix} = U$

$l_{11} = l_{22} = l_{33} = 1 \Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{7}{\alpha-3} & 1 \end{pmatrix}$

$\Rightarrow A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{7}{\alpha-3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha-3 & 5 \\ 0 & 0 & 3-\frac{35}{\alpha-3} \end{pmatrix}$

$|A| = |LU| = |L| \cdot |U| = (1)(1)(1) \cdot (1)(\alpha-3)\left(3 - \frac{35}{\alpha-3}\right)$

$= 3\alpha - 44$

Now we have two cases:

$\boxed{\text{(i) case (1)}}$: If $|A| \neq 0 \Rightarrow \alpha \neq \frac{44}{3}$ we get unique solution

$Ly = \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{7}{\alpha-3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix} \Rightarrow y_1 = 1 \Rightarrow y_2 = 5 \Rightarrow y_3 = 3 - \frac{35}{\alpha-3}$

$\Rightarrow y = \begin{bmatrix} 1 \\ 5 \\ 3-\frac{35}{\alpha-3} \end{bmatrix} \Rightarrow Ux = y \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha-3 & 5 \\ 0 & 0 & 3-\frac{35}{\alpha-3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3-\frac{35}{\alpha-3} \end{pmatrix}$

$\Rightarrow x_3 = 1 \Rightarrow x_2 = 0 \Rightarrow x_1 = 1 \Rightarrow X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ solution of the system

$\boxed{\text{(ii) Case(2)}}$: If $|A| = 0 \Rightarrow \alpha = \frac{44}{3}$ we get infinitely many solutions

So $|A| = 0$, *gives,* $\alpha = 44/3$ *and for this value of* α *we have non-trivial solutions. By solving the lower-triangular system of the form* $L\mathbf{y} = [1, 8, 3]^T$ *of the form*

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3/5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix} = \mathbf{b},$$

we obtained the solution $\mathbf{y} = [1, 5, 0]^T$. *Now solving the upper-triangular system* $U\mathbf{x} = \mathbf{y}$ *of the form*

$$U\mathbf{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 35/3 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = \mathbf{y}.$$

If we choose $x_3 = t \in R,\ t \neq 0$, *then,* $x_2 = 3(1-t)/7$ *and* $x_1 = (4+3t)/7$, *then the non-trivial solutions of the given system is* $\mathbf{x}^* = [(4+3t)/7, 3(1-t)/7, t]^T$. •