The Divergence Theorem

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The Divergence Theorem

Let Q be a solid region bounded by a closed surface S oriented by a normal vector directed outward and if $\mathbf{F}$ is vector field $\mathscr{C}^1$. Then

$$\underbrace{\iint_S \mathbf{F}.\mathbf{n}dS}_{\substack{\text{total outward flux} \\ \text{over the interior}}} = \underbrace{\iiint_Q \nabla.\mathbf{F}dV}_{\substack{\text{integral of local flux} \\ \text{through the surface } S}} = \iiint_Q \mathrm{div}\mathbf{F}dV. \quad (1)$$

Example

Compute the outward flux $\iint_S \mathbf{F}.\mathbf{n}dS$ of the vector field $\mathbf{F} = (yz - 3x)\mathbf{i} + (x - 2y)\mathbf{j} + (2 + z^2)\mathbf{k}$ through S, which is the surface of the ellipsoid $2x^2 + 2y^2 + z^2 = 8$ lying above the plane $z = 0$.

Solution The surface S is not closed (is not the boundary of the considered solid), so we cannot use divergence theorem.
Add a second surface S' so that $S \cup S'$ is a closed surface with interior D. We can take the surface S' the disc $x^2 + y^2 \leq 4$ in the $xy-$plane

$$\iint_S \mathbf{F}.\mathbf{n}dS + \iint_{S'} \mathbf{F}.\mathbf{n}'dS = \iiint_D \mathrm{div}\mathbf{F}dV.$$

Hence

$$\iint_S \mathbf{F}.\mathbf{n}dS = \iiint_D \mathrm{div}\mathbf{F}dV - \iint_{S'} \mathbf{F}.\mathbf{n}'dS.$$

$\mathrm{div}\mathbf{F} = -5 + 2z$,

$$\begin{aligned}
\iiint_D \mathrm{div}\mathbf{F}dV &= \iiint_D (-5+2z)dV \\
&= \int_0^{2\pi}\int_0^2\int_0^{\sqrt{8-2r^2}} (-5+2z)rdzdrd\theta \\
&= 2\pi\int_0^2 (-5(8-2r^2)+(8-2r^2)^2)rdr \\
&= 2\pi\int_0^2 (24r-22r^3+4r^5)dr = 64\pi.
\end{aligned}$$

$$
\begin{aligned}
-\iint_{S'} \mathbf{F}.\mathbf{n}' dS &= -\iint_{S'} \mathbf{F}.(-\mathbf{k}) dS = \iint_{S'} (2+z^2) dS \\
&= \iint_{S'} 2 dS = 8\pi.
\end{aligned}
$$

Hence

$$
\iint_{S} \mathbf{F}.\mathbf{n} dS = 64\pi + 8\pi = 72\pi.
$$

Example

Use the Divergence Theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = (x^3, y^3, z^3)$, where S is the surface of a solid bounded by the cone $x^2 + y^2 - z^2 = 0$ and the plane $z = 1$.

Solution:

Applying the Divergence Theorem, we can write:

$$\begin{aligned} I &= \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_G (\nabla\cdot\mathbf{F})\, dV \\ &= \iiint_G \left[\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^3\right)\right] dxdydz \\ &= 3\iiint_G \left(x^2+y^2+z^2\right) dxdydz. \end{aligned}$$

By changing to cylindrical coordinates, we have

$$\begin{aligned} I &= 3\iiint\limits_G \left(x^2+y^2+z^2\right) dxdydz \\ &= 3\int_0^1 dz \int_0^{2\pi} d\varphi \int_0^z \left(r^2+z^2\right) rdr = 6\pi\int_0^1 \left[\left(\frac{r^4}{4}+\frac{z^2r^2}{2}\right)\Big|_{r=0}^{z}\right] dz \\ &= 6\pi\int_0^1 \frac{3z^4}{4} dz = \frac{9\pi}{2}\left[\left(\frac{z^5}{5}\right)\Big|_0^1\right] = \frac{9\pi}{10}. \end{aligned}$$

Example

Evaluate the surface integral $\iint\limits_S x^3 dydz + y^3 dxdz + z^3 dxdy$,

where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ that has upward orientation.

Solution:

Using the Divergence Theorem, we can write:

$$\begin{aligned} I &= \iint\limits_{S} x^3 dydz + y^3 dxdz + z^3 dxdy = \iiint\limits_{G} \left(3x^2 + 3y^2 + 3z^2\right) dxdyd \\ &= 3 \iiint\limits_{G} \left(x^2 + y^2 + z^2\right) dxdydz. \end{aligned}$$

By changing to spherical coordinates, we have

$$\begin{aligned}
I &= 3\iiint\limits_G \left(x^2+y^2+z^2\right) dxdydz = 3\iiint\limits_G r^2\cdot r^2 \sin\theta drd\psi d\theta \\
&= 3\int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta \int_0^{a} r^4 dr \\
&= 3\cdot 2\pi\cdot\left[(-\cos\theta)|_0^{\pi}\right]\cdot\left[\left(\frac{r^5}{5}\right)\Big|_0^{a}\right] = \frac{12\pi a^5}{5}.
\end{aligned}$$

Example

Using the Divergence Theorem calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = (2xy, 8xz, 4yz)$, where is the surface of tetrahedron with vertices $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 1, 0)$, $D = (0, 0, 1)$.

Solution:

By Divergence Theorem,

$$\begin{aligned} I &= \iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_G (\nabla \cdot \mathbf{F})\, dV \\ &= \iiint\limits_G \left[\frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(8xz) + \frac{\partial}{\partial z}(4yz)\right] dV \\ &= \iiint\limits_G (2y + 0 + 4y)\, dxdydz = 6 \iiint\limits_G ydxdydz. \end{aligned}$$

$$
\begin{aligned}
I &= 6\iiint\limits_{G} y\,dx\,dy\,dz = 6\int_{0}^{1} dx \int_{0}^{1-x} dy \int_{0}^{1-x-y} y\,dz \\
&= 6\int_{0}^{1} dx \int_{0}^{1-x} (1-x-y)\,y\,dy = 6\int_{0}^{1} dx \int_{0}^{1-x} \left[y\,(1-x)-y^{2}\right] dy \\
&= 6\int_{0}^{1} \left[\left((1-x)\,\frac{y^{2}}{2}-\frac{y^{3}}{3}\right)\Bigg|_{y=0}^{1-x}\right] dx \\
&= 6\int_{0}^{1} \left[\frac{(1-x)^{3}}{2}-\frac{(1-x)^{3}}{3}\right] dx \\
&= 6\cdot\frac{1}{6}\int_{0}^{1} (1-x)^{3}dx = \frac{1}{4}.
\end{aligned}
$$

Example

Use the Divergence Theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x, y, z) = (x, y, z)$, where S is the surface of the solid bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = -1$ and $z = 1$.

Solution:

Using the Divergence Theorem, we can have:

$$\begin{aligned}
\iint\limits_S \mathbf{F}\cdot d\mathbf{S} &= \iiint\limits_G (\nabla\cdot\mathbf{F})\,dV \\
&= \iiint\limits_G \left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)\right] dxdydz \\
&= \iiint\limits_G (1+1+1)\,dxdydz = 3\iiint\limits_G dxdydz.
\end{aligned}$$

By switching to cylindrical coordinates, we have

$$
\begin{aligned}
I &= 3\iiint\limits_{G} dxdydz = 3\int\limits_{-1}^{1} dz \int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{a} rdr \\
&= 3\cdot 2\cdot 2\pi\cdot\left[\left(\frac{r^2}{2}\right)\Big|_0^a\right] = 6\pi a^2.
\end{aligned}
$$