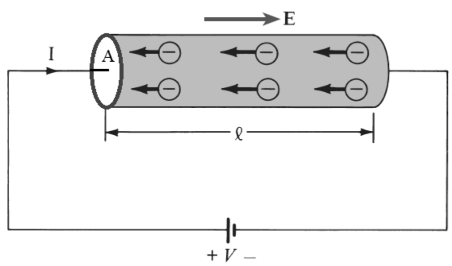
Chapter 4: Direct-Current Circuits

4.1 Conduction of electricity

Under static conditions, the electric field is zero inside a conductor. However, if an electric field is maintained by an external source (for example, battery) then charge carriers drift in the field and there is an electric current.

Suppose charges are moving perpendicular to a surface of area A as shown in the figure below



The current is defined as the rate at which charge flows through this surface.
The SI unit of current is the ampere (A):

$$1A = \frac{1C}{1s}$$

Thus, in a current of one ampere (1A), charge is being transferred at a rate of one coulomb per second.

The average current is:

$$I_{av} = \frac{\Delta Q}{\Delta t}$$

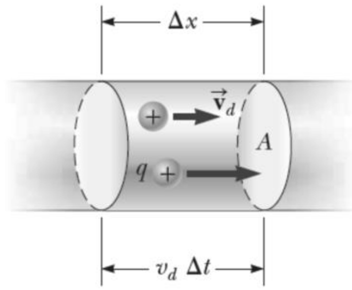
The instantaneous current

$$I = \frac{dQ}{dt}$$

The direction of the current is opposite the direction of flow of electrons

4.1.1 Microscopic model of current

Consider the current in a cylindrical conductor of cross-sectional area A and the direction of the applied electric field is to the right that cause the charge carriers to move in the x direction with Δx with a drift velocity υ_d



$$dQ = (nAv_d\,dt)q$$

$$I = \frac{dQ}{dt} = nqv_dA$$

n represents the charge carrier density (the number of mobile charge carriers per unit volume)

Example 4.1

A copper wire has a cross-sectional area of 3.31×10^{-6} m^2. It carries a constant current of 10 A. What is the drift speed of the electrons in the wire? Assume each copper atom contributes one free electron to the current. The density of copper is 8.92 g/cm^3

$$v_d = \frac{I}{nqA} \qquad n = \frac{N_a}{V} \qquad V = \frac{M}{\rho}$$

N_a is Avogadro's number $= 6.02\times10^{23}\ mol^{-1}$

M is molar mass of copper $= 63.5\ g/mol$

$$\Rightarrow v_d = \frac{IM}{qAN_A\rho} = \frac{(10\,A)(63.5\times10^{-3}\ Kg/mol)}{(1.6\times10^{-19}\,C)(3.31\times10^{-6}\,m^2)(6.02\times10^{23}\,mol^{-1})(8.92\times10^{3}\ Kg/m^3)}$$

$$\Rightarrow v_d = 2.23\times10^{-4}\ m/s$$

4.1.2 Current density, conductivity, resistivity and mobility

The current density **J** in the conductor is defined as the current per unit area:

$$J = \frac{\Delta I}{\Delta A} = nqv_d$$

The unit of J is A/m^2

If the current density is not normal to the surface

$$\Delta I = J \cdot \Delta A$$

Or in general:

$$\Delta I = \int_s J \cdot dA$$

For ordinary conductors, the current density J is proportional to the electric field intensity E:

$$J = \sigma E$$

where σ is the conductivity of the material and it is expressed in amperes per volt per meter **A/(V.m)** , or in siemens per meter (S/m)
The conductivity of metals generally increases with decrease in temperature.

If the electric field is uniform within a straight wire of uniform cross-sectional area A and length ℓ:

$$J = \sigma E = \sigma \frac{\Delta V}{\ell}$$

$$\Delta V = \frac{\ell}{\sigma} J = \left(\frac{\ell}{\sigma A} \right) I$$

The ratio of the potential difference across a conductor to the current passing through it can be defined as the resistance:

$$R = \frac{\Delta V}{I}$$

And this Its unit is ohm

$$1\Omega = \frac{1V}{1A}$$

The inverse of conductivity is resistivity:

$$\rho = \frac{1}{\sigma}$$

Thus, the resistance of a wire can be written as:

$$R = \frac{\ell}{\sigma A} = \rho \frac{\ell}{A}$$

The drift velocity υ_d is proportional to the electric field intensity E

$$v_d = \mu E$$

Where μ is the mobility of carriers and from the previous relations we can write the mobility to be:

$$\mu = \frac{\sigma}{nq}$$

Example 4.2

A wire of diameter 1 mm and conductivity 5×10^7 S/m has 10^{29} free electrons per cubic meter when an electric field of 10 mV/m is applied. Determine:

(a) The charge density of free electrons
(b) The current density
(c) The current in the wire
(d) The drift speed of the electrons

a) $\rho = nq = (10^{29})(-1.6 \times 10^{-19}) = -1.6 \times 10^{10}\ C/m^3$

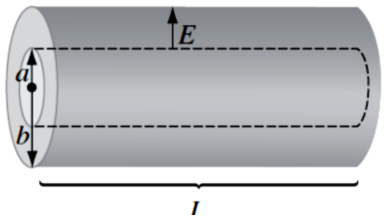
b) $J = \sigma E = (5 \times 10^{7})(10 \times 10^{-3}) = 500\ KA/m^2$

c) $I = JA = (500 \times 10^{3})\left(\pi \left(\frac{1 \times 10^{-3}}{2}\right)^2\right) = 0.393\ A$

d) $J = nq\upsilon_d \Rightarrow \upsilon_d = \frac{J}{nq} = \frac{J}{\rho} = \frac{500 \times 10^{3}}{1.6 \times 10^{10}} = 3.125 \times 10^{-5}$ m/s

Example 4.3

Two long coaxial metal cylinders (radii a and b) are separated by material of conductivity σ. If they are maintained at a potential difference V, what current flows from one to the other, in a length L?



from Gauss's law the electric field between the cylinders is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon r}\hat{n}$$

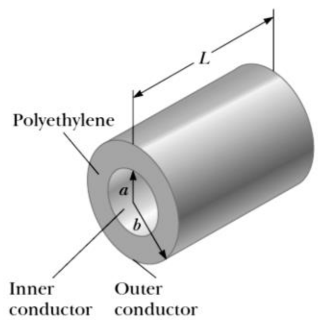
$$I = \int \vec{J}\cdot d\vec{a} = \sigma \int \vec{E}\cdot d\vec{a} = \frac{\sigma \lambda L}{\epsilon}$$

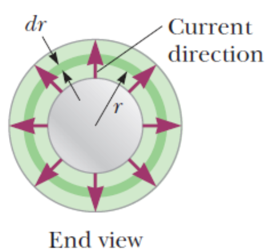
$$V = -\int_b^a \vec{E}\cdot d\vec{l} = \frac{\lambda}{2\pi\epsilon}\ln\left(\frac{b}{a}\right)$$

$$R = \frac{V}{I} = \frac{\left(\frac{\lambda}{2\pi\epsilon}\ln\left(\frac{b}{a}\right)\right)}{\left(\frac{\sigma\lambda L}{\epsilon}\right)} \Rightarrow \boxed{R = \frac{\ln(b/a)}{2\pi\sigma L}}$$

Example 4.4

A coaxial cable consists of two concentric cylindrical conductors. The region between the conductors is completely filled with polyethylene plastic as shown in the figure below. If the radius of the inner conductor is a = 0.5 cm and the radius of the outer conductor is b = 1.75 cm, and the length of the cable is L = 15 cm. Calculate the resistance of the plastic between the two conductors if the resistivity of the plastic is $1.0 \times 10^{13}\ \Omega \cdot m$.





$$dR = \frac{\rho\, dr}{A} \Rightarrow R = \int_a^b \frac{\rho\, dr}{2\pi r L}$$

$$R = \frac{\rho}{2\pi L} \ln\left(\frac{b}{a}\right)$$

$$R = \frac{1 \times 10^{13}}{2\pi (0.15)} \ln\left(\frac{1.75}{0.5}\right)$$

$$= 1.33 \times 10^{13}\ \Omega$$

4.2 Electrical power

The rate at which the electric potential energy of the system decreases as the charge q passes through a resistor can be written as:

$$dU = Vdq$$

$$dU = VIdt$$

$$\Rightarrow U = VIt$$

As the power P represents the rate at which energy is delivered to a system

$$P = VI$$

If we assume all electric potential is consumed in the resistor as heat:

$$P=I^2R$$

The thermal energy H is:

$$H=I^2Rt$$

The process by which energy is transformed to internal energy in a conductor of resistance R is often called joule heating and this thermal energy can be measured in joules or calories: 1Cal = 4.2 J

Example 4.5

An electric heater is constructed by applying a potential difference of 120 V across a Nichrome wire that has a total resistance of 8 **Ω**. Find the current carried by the wire and the power rating of the heater.

$$I = \frac{\Delta V}{R} = \frac{120}{8} = 15\ A$$

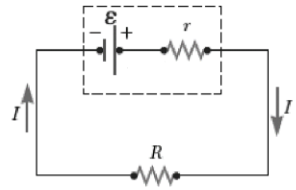
$$P = I^2 R = (15)^2 (8) = 1.8 \times 10^3\ W$$

4.3 Electromotive force

Consider a conductor whose ends are maintained at a potential difference V using battery as a source of energy.
Because the potential difference at the battery terminals is constant in a particular circuit, the current in the circuit is constant in magnitude and direction and is called **direct current** DC

A battery is called a source of electromotive force (emf) $\boldsymbol{\varepsilon}$.
$\boldsymbol{\varepsilon}$ of a battery is the maximum possible voltage the battery can provide between its terminals.

However, real batteries have a certain internal resistance, r , and the potential difference between their terminals is $\boldsymbol{\varepsilon} - Ir$, (**when a current I is flowing**)



Therefore,

$$\Delta V = \varepsilon - Ir$$

ΔV equals the potential difference across the external resistance R (called the load resistance)

$$\varepsilon = IR + Ir$$

$$I = \frac{\varepsilon}{R + r}$$

Example 4.6

A battery has an emf of 15 V. The terminal voltage of the battery is 11.6 V when it is delivering 20 W of power to an external load resistor R.
(a) What is the value of R?
(b) What is the internal resistance of the battery?

a) $R = \frac{(\Delta V)^2}{P} = \frac{(11.6)^2}{20} = 6.73\,\Omega$

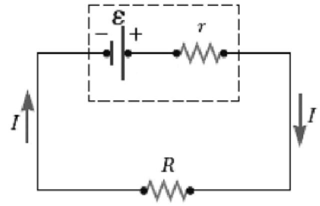
b) $\varepsilon = IR + Ir \Rightarrow r = \frac{\varepsilon - IR}{I}$

$\because I = \frac{\Delta V}{R} \Rightarrow r = \frac{(\varepsilon - \Delta V)R}{\Delta V}$

$\Rightarrow r = \frac{(15 - 11.6)(6.73)}{11.6} = 1.97\,\Omega$

Example 4.7

Find the load resistance R for which the maximum power is delivered to the load resistance in the circuit below.



$$P = I^2 R = \frac{\varepsilon^2 R}{(R+r)^2}$$

$$\frac{dP}{dR} = \frac{d}{dR}\left[\varepsilon^2 R (R+r)^{-2}\right] = 0$$

$$\Rightarrow \varepsilon^2 (R+r)^{-2} + \varepsilon^2 R(-2)(R+r)^{-3} = 0$$

$$\cancel{\varepsilon^2}\,(R+r)^{-2} = 2\cancel{\varepsilon^2} R\,(R+r)^{-3}$$

$$\frac{R}{R+r} = \frac{1}{2} \Rightarrow 2R = R + r$$

$$\Rightarrow \boxed{R = r}$$

$$P_{max} = \frac{\varepsilon^2}{4r}$$