Introduction to Real Analysis

Differentiation

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Derivative

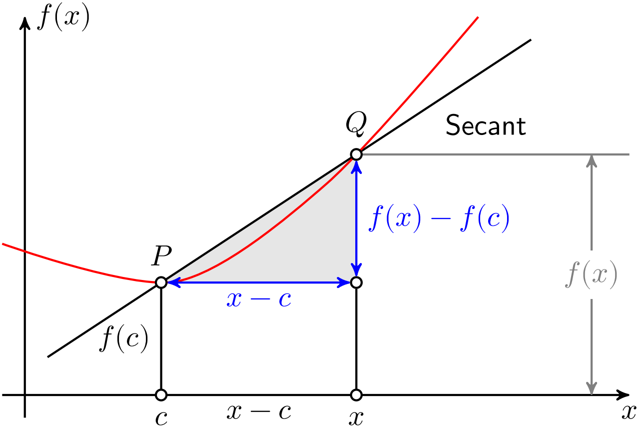
Definition

Let $f : I \to \mathbb{R}$ (where I is an interval) and $c \in I$ then if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists, it is called the derivative of f at c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$


$f(x)$
Q
Secant
$f(x) - f(c)$
P
$x - c$
$f(x)$
$f(c)$
c
$x - c$
x
x

Derivative

Examples

$f(x) = k$

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Derivative

Examples

$f(x) = x^n, n \in \mathbb{N}$

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Derivative

Examples

$f(x) = |x|$

.

If f is defined on $I = [a, b]$ then the derivatives at a and b are

$$f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

$$f'(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

If $f : (a, b) \to \mathbb{R}$ is differentiable then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative

Examples

$f(x) = \sin x$

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Derivative

Theorem

If the function $f : I \longrightarrow \mathbb{R}$ is differentiable at $c \in I$, then it is continuous at c.

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Derivative

Theorem

If the functions $f, g : I \to \mathbb{R}$ are differentiable at $c \in I$, then

1. $f+g$ is differentiable at c and
$$(g+f)'(c) = g'(c) + f'(c)$$
2. fg is differentiable at c and
$$(fg)'(c) = f(c)g'(c) + f'(c)g(c)$$
3. If $g(c) \neq 0$ then $\frac{f}{g}$ is differentiable at c and
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$$

If the function $f : I \longrightarrow \mathbb{R}$ is differentiable at $c \in I$ then

1. f^2 is differentiable at c and

$$(f^2)'(c) = 2f(c)f'(c)$$

2. f^n is differentiable at c and

$$(f^n)'(c) = nf^{n-1}(c)f'(c)$$

Derivative

Examples

1. x^n
2. $p(x)$
3. x^{-n}

.

Derivative

Theorem

Let I, J be intervals, $f : I \to \mathbb{R}$ and $f(I) \subset J$, $g : J \to \mathbb{R}$. If f is differentiable at $c \in I$ and g is differentiable at $f(c)$ then $gof : I \to \mathbb{R}$ is differentiable at c and

$$(gof)'(c) = g'(f(c)) \cdot f'(c)$$

If

$$y = f(x), \qquad w = g(y)$$

then

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx}$$

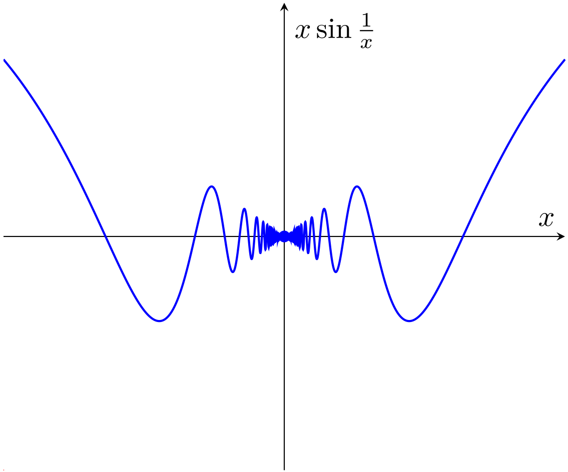
Derivative

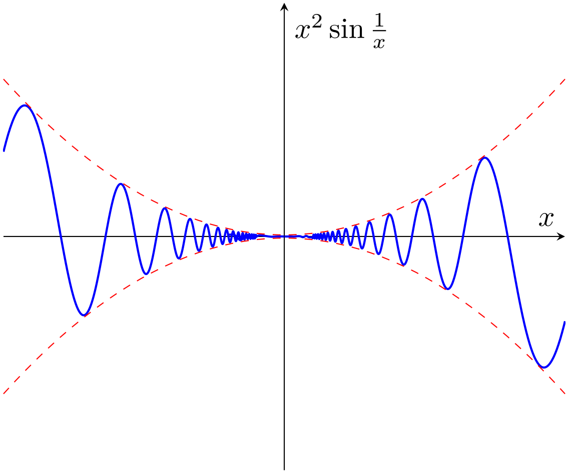
Examples

1.
$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

2.
$$g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

3.
$$h(x) = \begin{cases} x^2\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$


$x \sin \frac{1}{x}$
x


$x^2 \sin \frac{1}{x}$
x

Derivative

Theorem

If $f : I \to \mathbb{R}$ is injective and continuous on the interval I and if f is differentiable at $c \in I$ then f^{-1} is differentiable at $d = f(c)$ iff $f'(c) \neq 0$

$$(f^{-1})'(d) = \frac{1}{f'(c)}$$

or

$$(f^{-1})'(d) = \frac{1}{f'(f^{-1}(d))}$$

Derivative

Examples

The function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^3$$

is $1-1$ and differentiable, find

$$(f^{-1})'(8)$$

.

Derivative

Examples

The function $f : [-\pi/2, \pi/2] \to \mathbb{R}$

$$f(x) = \sin(x)$$

is $1-1$ and differentiable, find $(f^{-1})'(x)$

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Local Extrema

Definition

The function $f : D \to \mathbb{R}$ has a local maximum at $c \in D$ if there is a neighborhood $U = (c - \delta, c + \delta)$ such that

$$f(x) \leq f(c) \qquad \forall x \in U \cap D$$

and it has a local minimum at $c \in D$ if there is a neighborhood $U = (c - \delta, c + \delta)$ such that

$$f(x) \geq f(c) \qquad \forall x \in U \cap D$$

Extremum

Theorem

If f has an extremum on (a, b) at c and if f is differentiable at c then

$$f'(c) = 0$$

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Critical Point

Definition

c is a critical point of f if

1. f is not differentiable at c
2. or $f'(c) = 0$

If the function $f : (a, b) \to \mathbb{R}$ has a local extremem at c then c is a critical point of f

Extremum

Examples

$f : [-2, 3] \to \mathbb{R}$

$$f(x) = x^3 - 3x$$

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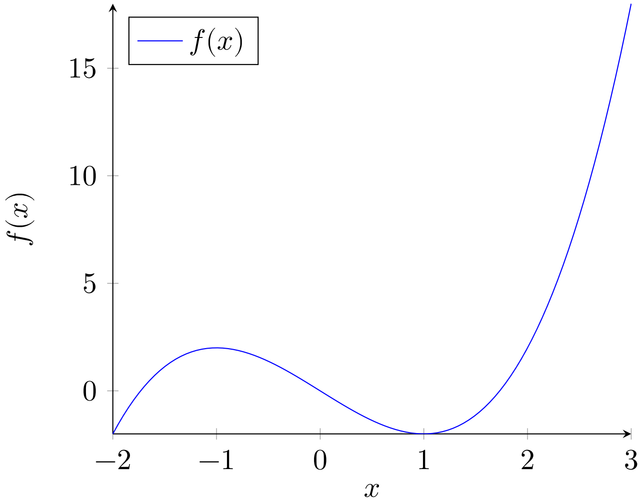
$f(x)$
15
10
5
0
-2
-1
0
1
2
3
x
$f(x)$

Rolle's Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ then there is $c \in (a, b)$ such that

$$f'(c) = 0$$

.

Rolle's Theorem

Examples

$f : [1, 5] \to \mathbb{R}$

$$f(x) = -x^2 + 6x - 6$$

.

Mean Value Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) then there is $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

.

Mean Value Theorem

Examples

$f : [0, 2] \to \mathbb{R}$

$$f(x) = x^3$$

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Mean Value Theorem

Examples

Prove that

$$\sin x \leq x \qquad \forall x \geq 0$$

.

Applications of Mean Value Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) then

1. If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$
2. If $f'(x) \neq 0$ for all $x \in (a, b)$ then f is injective on $[a, b]$

.

Applications of Mean Value Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) then

1. If $f'(x) \geq 0$ for all $x \in (a, b)$ then f is increasing on $[a, b]$
2. If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$
3. If $f'(x) \leq 0$ for all $x \in (a, b)$ then f is decreasing on $[a, b]$
4. If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$

.

First derivative test

Theorem

If $f : D \to \mathbb{R}$ is continuous and c is a critical point of f and there is an open interval $U \subset D$ which contains c such that

1. $$f'(x) < 0 \qquad \forall x \in U, x < c$$
   $$f'(x) > 0 \qquad \forall x \in U, x > c$$
   then $f(c)$ is a local minimum for f
2. $$f'(x) > 0 \qquad \forall x \in U, x < c$$
   $$f'(x) < 0 \qquad \forall x \in U, x > c$$
   then $f(c)$ is a local maximum for f

First derivative test

Theorem - continued

1. $f'(x)$ has the same sign on $x \in U - \{c\}$ then $f(c)$ is not a local extremum of f

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Darboux

Theorem

If $f : I = [a, b] \to \mathbb{R}$ is differentiable and λ is between $f'(a)$ and $f'(b)$, i.e.,

$$f'(a) < \lambda < f'(b) \qquad or \qquad f'(b) < \lambda < f'(a)$$

then there is $c \in (a, b)$ such that

$$f'(c) = \lambda$$

.

Darboux

Examples

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

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L'Hopital's Rule

Theorem

Let $f, g : I \to \mathbb{R}$ be continuous I and differentiable on $I - \{c\}$ where $c \in I$. If

1. $g'(x) \neq 0 \qquad \forall\, x \in I - \{c\}$
2. $f(c) = g(c) = 0$
3. the limit $\lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$ exists in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

L'Hopital's Rule

Theorem

Let $f, g : [a, \infty) \to \mathbb{R}$ be differentiable on $[a, \infty)$ and suppose that

1. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$
2. $g'(x) \neq 0 \qquad \forall\, x > a$
3. the limit $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists n $\bar{\mathbb{R}}$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

L'Hopital's Rule

Theorem

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable, then if

1. $g'(x) \neq 0 \qquad \forall\, x \in (a, b)$
2. $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = \infty$
3. the limit $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists in $\bar{\mathbb{R}}$

then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

L'Hopital's Rule

There are other indeterminate forms

$$\frac{\infty}{\infty} \qquad 1^{\infty} \qquad 0 \cdot \infty$$

$$\infty - \infty \qquad 0^{0} \qquad \infty^{0}$$

L'Hopital's Rule

Examples

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2}$$

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L'Hopital's Rule

Examples

$$\lim_{x\to\infty} \frac{\log(x)}{x}$$

.

L'Hopital's Rule

Examples

$$\lim_{x \to 0^+} (1 + \frac{3}{x})^x$$

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L'Hopital's Rule

Examples

$$\lim_{x\to\infty} \frac{\cos(x)+x}{x}$$

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Taylor's Theorem

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

Taylor's Theorem

Theorem

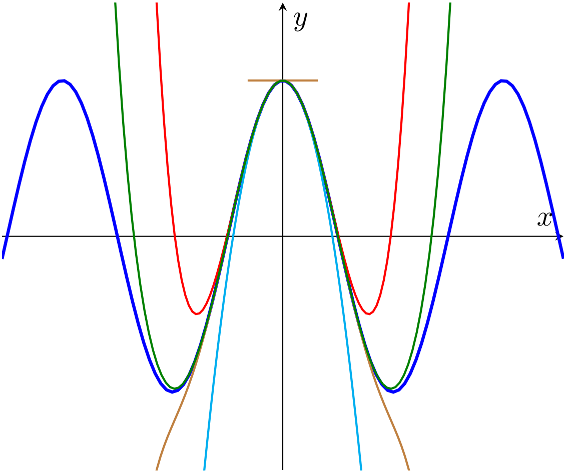
Let $f \in C^{n}[a,b]$ i.e., $f', \dots, f^{(n)}$ are continuous on $[a,b]$ and $f^{(n)}$ is differentiable on (a,b). If $x_0 \in [a,b]$ then for every $x \in [a,b] - \{x_0\}$ there is c between x_0 and x such that

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \dots$$

$$+\frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

.

$\cos x$



Taylor's Thorem

Examples

1. Approximate $f(x) = \sqrt{x+1}$ on $(-1,1)$ by a polynomial of degree 3 at $x_0 = 0$
2. What is the error in approximation on $[0, \frac{1}{2}]$

.

Taylor's Thorem

Examples

1. Approximate $f(x) = e^{x}$ by a polynomial at $x_0 = 0$
2. If we want to approximate the number e of error not exceeding 10^{-2} what is the minimum value of n?

.

Young's Theorem

Theorem

If $f, f', \dots, f^{(n)}$ are all continou $[a, b]$ and $f^{(n)}$ is differentiable at $x_0 \in [a, b]$ and if $x \in [a, b]$ then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \dots$$

$$+\frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1} + E$$

where $\frac{E}{(x - x_0)^{n+1}} \to 0$ as $x \to x_0$

.

Young's Theorem

Theorem

If

$$f'(c) = f''(c) \ldots, f^{(m-1)}(c) = 0$$

and

$$f^{(m)}(c) \neq 0$$

then

- If m is odd then $f(c)$ not a local extremum
- If m is even and $f^{(m)}(c) < 0$ then $f(c)$ is a local maximum
- If m is ever and $f^{(m)}(c) > 0$ then $f(c)$ is a local minimum

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Young's Theorem

Examples

Decide whether $f(0)$ is an extremum value of f

$$f(x) = x \sin x$$

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Exercises

1. Use the definition to find the derivative of $f(x) = \dfrac{1}{x}, \; x \neq 0$
2. Find the set of points where the function f is not differentiable
    1. $f(x) = |x^2 - 1|$
    2. $f(x) = x|x|$
3. If $g(0) = g'(0) = 0$, find $f'(0)$ where

$$f(x) = \begin{cases} g(x)\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Exercises

1. Let
$$f\left(x\right)=\begin{cases} x^2 & x\in\mathbb{Q} \\ 0 & x\in\mathbb{Q}^c \end{cases}$$
Prove that f is differentiable at $x=0$, and evaluate $f'(0)$.
2. If the function f satisfies $|f(x)|\leq|x|^r$, where $r>1$, prove that f is differentiable at $x=0$.
3. Let $f:\mathbb{R}\to\mathbb{R}$. The function f is even if $f(-x)=f(x)$ for all $x\in\mathbb{R}$, and odd if $f(-x)=-f(x)$ for all $x\in R$. If f is differentiable, prove that f' is odd when f is even, and even when f is odd.

Exercises

1. Use the definition to show that $f(x) = \sqrt{x^2+1}$ is differentiable on $\mathbb{R}$, then prove that there is $c \in (0,1)$ such that
$$\sqrt{2} - 1 = \frac{c}{\sqrt{c^2+1}}$$
2. If $f : \mathbb{R} \to \mathbb{R}$, and there is a real constant $K > 0$ such that
$$|f(x) - f(y)| \leq K|x-y|^2 \quad \forall x, y \in \mathbb{R}$$
show that f is constant.
3. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
4. Prove that
$$\sqrt{1+x} < 1 + \frac{x}{2} \quad \forall\ x > 0$$

Exercises

1. Evaluate the following limits
   1. $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$
   2. $\lim_{x\to 0^+} \left(1 + \frac{2}{x}\right)^x$
2. Let $g \in C^2(\mathbb{R})$ such that $g(0) = g'(0) = 0$ and $g''(0) = 6$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and defined by
$$f(x) = \frac{g(x)}{x}, \qquad x \neq 0$$
   Find $f(0)$, and discuss the differentiability of f at $x = 0$.
3. If $f(x) = x^2 \sin(\frac{1}{x})$, $g(x) = \sin x$, show that $\lim_{x\to 0} \frac{f(x)}{g(x)}$ exists, while $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

Exercises

1. Prove that for $x > 0$,
$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{x+1} \leq 1 + \frac{x}{2}$$
2. Decide whether $f(0)$ is and extremum value of $f(x) = \sin x - x + \frac{x^3}{6}$.