

Lecture 1: Separable Equations

Separable equation.

A differential equation of the form

$$\varphi(y) dy + \psi(x) dx = 0$$

is called a **separable equation**.

Standard separable ODE.

An equation of the form

$$y' = \varphi(x) \psi(y)$$

is also called an equation with separable variables. Separating variables gives

$$\frac{dy}{\psi(y)} = \varphi(x) dx.$$

Example 1.

Given equation:

$$x^2 y^2 y' = y - 1 \quad \left(\text{equivalently } y' = \frac{y-1}{x^2 y^2} \right).$$

Divide the equation by $x^2(y-1) \neq 0$:

$$\frac{y^2 y'}{y-1} = \frac{1}{x^2} \implies \frac{y^2}{y-1} dy = \frac{dx}{x^2}.$$

Integrating:

$$\int \frac{y^2}{y-1} dy = \int \frac{1}{x^2} dx.$$

Rewrite:

$$\frac{y^2}{y-1} = y + 1 + \frac{1}{y-1}.$$

Hence:

$$\int \frac{y^2}{y-1} dy = \int \left(y + 1 + \frac{1}{y-1} \right) dy = \frac{y^2}{2} + y + \ln |y-1|.$$

And:

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = -\frac{1}{x}.$$

So the implicit general solution is

$$\frac{y^2}{2} + y + \ln |y-1| = -\frac{1}{x} + C.$$

$$\frac{y^2}{2} + y + \ln |y-1| = -\frac{1}{x} + 5,$$

together with the special solution $y = 1$.

Example 2.

Given equation:

$$e^y(1+x^2) dy - 2x(1+e^y) dx = 0.$$

Rewrite the equation as

$$e^y(1+x^2) dy = 2x(1+e^y) dx.$$

Hence

$$\frac{e^y}{1+e^y} dy = \frac{2x}{1+x^2} dx.$$

Integrate both sides. We obtain

$$\int \frac{e^y}{1+e^y} dy = \int \frac{2x}{1+x^2} dx.$$

For the left-hand side, let

$$u = 1 + e^y, \quad du = e^y dy,$$

so that

$$\int \frac{e^y}{1+e^y} dy = \int \frac{du}{u} = \ln(1+e^y).$$

For the right-hand side,

$$\int \frac{2x}{1+x^2} dx = \ln(1+x^2).$$

Thus

$$\ln(1+e^y) = \ln(1+x^2) + C.$$

Exponentiating, we get

$$1+e^y = C(1+x^2),$$

hence

$$e^y = C(1+x^2) - 1,$$

and therefore

$$\boxed{y = \ln(C(1+x^2) - 1)}.$$

$$\boxed{\ln(1+e^y) = \ln(1+x^2) + C}$$

or equivalently

$$\boxed{y = \ln(C(1+x^2) - 1)}.$$

Example 3: Solve $y' = \cos(x - y - 1)$.

Consider

$$y' = \cos(x - y - 1).$$

Use the substitution $z = x - y - 1$. Then

$$z' = 1 - y'.$$

Since $y' = \cos z$, we have

$$z' = 1 - \cos z.$$

Separate variables and integrate:

$$\int \frac{dz}{1 - \cos z} = \int dx.$$

Hence,

$$-\cot\left(\frac{z}{2}\right) = x + C.$$

Substitute back $z = x - y - 1$ to obtain the general implicit solution:

$$-\cot\left(\frac{x - y - 1}{2}\right) = x + C.$$

Lecture 2: Homogeneous Equations

Homogeneous function.

A function $f(x, y)$ is called a **homogeneous function of degree k** if

$$f(tx, ty) = t^k f(x, y)$$

for all t for which both sides are defined.

Homogeneous differential equation.

A differential equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is called **homogeneous** if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the **same** degree.

Equivalent form.

Homogeneous equations can be written as

$$y' = f(x, y),$$

where $f(x, y)$ is homogeneous of degree 0, i.e.

$$f(tx, ty) = f(x, y).$$

Equivalently, $f(x, y)$ depends only on the ratio $\frac{y}{x}$ (or $\frac{x}{y}$, when appropriate).

Method (reduction to separable variables).

A homogeneous equation can be transformed into an equation with separable variables by the substitution

$$y = ux, \quad u = u(x),$$

where u is a new unknown function. (One may also use the substitution $x = uy$.)

Example 1. Solve the equation

$$x dy = (x + y) dx.$$

Solution. This equation is homogeneous. Set $y = ux$. Then

$$dy = u dx + x du.$$

Substitute into the equation:

$$x(u dx + x du) = (x + ux) dx.$$

Simplifying gives

$$xu dx + x^2 du = x(1 + u) dx \implies x^2 du = x dx \implies x du = dx.$$

Now solve the separable equation:

$$du = \frac{dx}{x} \implies u = \ln|x| + C.$$

Returning to y , we obtain

$$y = xu = x(\ln|x| + C).$$

Note. During the reduction we effectively divided by x , so the solution $x = 0$ may be lost. Indeed, $x = 0$ is also a (singular) solution of the original differential equation.

Example 2. Solve the equation

$$x^2 dy = (y^2 - xy + x^2) dx$$

Solution:

Rewrite the equation as

$$\frac{dy}{dx} = \frac{y^2 - xy + x^2}{x^2} = \left(\frac{y}{x}\right)^2 - \frac{y}{x} + 1.$$

Let $y = vx$, then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}.$$

Substitute:

$$v + x \frac{dv}{dx} = v^2 - v + 1.$$

So,

$$x \frac{dv}{dx} = (v - 1)^2.$$

Separate variables:

$$\frac{dv}{(v - 1)^2} = \frac{dx}{x}.$$

Integrate:

$$-\frac{1}{v - 1} = \ln|x| + C.$$

Substitute back $v = \frac{y}{x}$:

$$-\frac{1}{\frac{y}{x} - 1} = \ln|x| + C.$$

Simplify:

$$\frac{x}{x - y} = \ln|x| + C.$$

Final Answer:

$$\boxed{\frac{x}{x - y} = \ln|x| + C}$$

Also, the singular solution is

$$\boxed{y = x}.$$

Lecture 3: Exact Equations

General form.

A differential equation is exact if

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

and there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N.$$

Then the solution is

$$\Psi(x, y) = C.$$

To test for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

How to solve.

Find $u(x, y)$ such that $u_x = M$ and $u_y = N$. Then the general implicit solution is

$$u(x, y) = C,$$

where C is an arbitrary constant.

Example 1.

$$2xy - 9x^2 + (2y + x^2 + 1) \frac{dy}{dx} = 0.$$

Let

$$\Psi(x, y) = y^2 + (x^2 + 1)y - 3x^3.$$

Then

$$\frac{\partial \Psi}{\partial x} = 2xy - 9x^2, \quad \frac{\partial \Psi}{\partial y} = 2y + x^2 + 1,$$

so the equation becomes

$$\frac{d}{dx} [\Psi(x, y(x))] = 0 \quad \Rightarrow \quad \Psi(x, y) = C.$$

Implicit solution:

$$y^2 + (x^2 + 1)y - 3x^3 = C.$$

Example 2 (with initial condition).

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3.$$

Here

$$M = 2xy - 9x^2 \Rightarrow M_y = 2x, \quad N = 2y + x^2 + 1 \Rightarrow N_x = 2x,$$

so the equation is exact.

Integrate M with respect to x :

$$\Psi(x, y) = \int (2xy - 9x^2) dx = x^2y - 3x^3 + h(y).$$

Differentiate with respect to y :

$$\frac{\partial \Psi}{\partial y} = x^2 + h'(y) = 2y + x^2 + 1 \Rightarrow h'(y) = 2y + 1 \Rightarrow h(y) = y^2 + y.$$

So

$$\Psi(x, y) = x^2y - 3x^3 + y^2 + y.$$

Implicit solution:

$$x^2y - 3x^3 + y^2 + y = C.$$

Apply $y(0) = -3$:

$$0 + 9 - 3 = C \Rightarrow C = 6.$$

Thus

$$y^2 + (x^2 + 1)y - 3x^3 = 6.$$

Example 3.

$$2xy^2 + 4 = 2(3 - x^2y) y', \quad y(-1) = 8.$$

Rewrite as

$$2xy^2 + 4 + 2(x^2y - 3)y' = 0.$$

Then

$$M = 2xy^2 + 4, \quad N = 2x^2y - 6,$$

and

$$M_y = 4xy, \quad N_x = 4xy,$$

so the equation is exact.

Integrate N with respect to y :

$$\Psi(x, y) = \int (2x^2y - 6) dy = x^2y^2 - 6y + h(x).$$

Differentiate with respect to x :

$$\Psi_x = 2xy^2 + h'(x) = 2xy^2 + 4 \Rightarrow h'(x) = 4 \Rightarrow h(x) = 4x.$$

Thus

$$\Psi(x, y) = x^2y^2 - 6y + 4x, \quad x^2y^2 - 6y + 4x = C.$$

Apply $y(-1) = 8$:

$$64 - 48 - 4 = C \Rightarrow C = 12.$$

So

$$x^2y^2 - 6y + 4x - 12 = 0.$$

Solving for y :

$$y = 3 \pm \frac{\sqrt{9 + 12x^2 - 4x^3}}{x^2}.$$

Choose the sign consistent with the initial condition.

Example 4

Solve

$$(2xy + y^2) dx + (x^2 + 2xy) dy = 0.$$

Here

$$M = 2xy + y^2, \quad N = x^2 + 2xy.$$

Compute

$$M_y = 2x + 2y, \quad N_x = 2x + 2y.$$

Thus $M_y = N_x$, so the equation is exact.

Solution:

We seek a function $F(x, y)$ such that

$$dF = M dx + N dy.$$

So,

$$F_x = 2xy + y^2.$$

Integrate with respect to x :

$$F(x, y) = \int (2xy + y^2) dx = x^2y + xy^2 + \phi(y),$$

where $\phi(y)$ is a function of y only.

Now differentiate with respect to y :

$$F_y = x^2 + 2xy + \phi'(y).$$

But $F_y = N = x^2 + 2xy$, so

$$\phi'(y) = 0.$$

Hence

$$\phi(y) = C_1,$$

a constant, and therefore

$$F(x, y) = x^2y + xy^2.$$

So the required solution is

$$F(x, y) = C,$$

that is,

$$\boxed{x^2y + xy^2 = C.}$$

Equivalently,

$$\boxed{xy(x + y) = C.}$$

Lecture 4: Non-Exact Differential Equations

Definition.

A first-order differential equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is called **exact** if there exists a function $F(x, y)$ such that

$$F_x = M(x, y), \quad F_y = N(x, y).$$

If

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

then the equation is **non-exact**.

Main idea.

Even when the equation is not exact, it may be possible to multiply it by a function $\mu(x, y)$, called an **integrating factor**, so that

$$\mu M dx + \mu N dy = 0$$

becomes exact.

Common cases.

1. If

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

depends only on x , then an integrating factor $\mu(x)$ exists and can be taken as

$$\mu(x) = \exp\left(\int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx\right).$$

2. If

$$\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

depends only on y , then an integrating factor $\mu(y)$ exists and can be taken as

$$\mu(y) = \exp\left(\int \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\right).$$

Summary.

- Exact equations satisfy $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- A non-exact equation may become exact after multiplying by an integrating factor $\mu(x)$ or $\mu(y)$.
- If no simple integrating factor exists, try alternative methods (e.g. homogeneous or separable substitutions).

Example 1. Integrating factor depending on y .

Solve

$$y dx + (xy + y^2) dy = 0.$$

Solution:

$$M = y, \quad N = xy + y^2.$$

$$M_y = 1, \quad N_x = y.$$

$$\frac{1}{M}(N_x - M_y) = \frac{y - 1}{y} = 1 - \frac{1}{y}$$

depends only on y , so

$$\mu(y) = \exp\left(\int \left(1 - \frac{1}{y}\right) dy\right) = \frac{e^y}{y}.$$

Multiply:

$$e^y dx + e^y(x + y) dy = 0.$$

Now exact:

$$F_x = e^y \Rightarrow F = xe^y + \phi(y).$$

$$F_y = xe^y + \phi'(y) = e^y(x + y) \Rightarrow \phi'(y) = ye^y.$$

$$\phi(y) = e^y(y - 1).$$

$$F = e^y(x + y - 1).$$

$$\boxed{e^y(x + y - 1) = C.}$$

Example 2. Integrating factor depending on x .

Solve

$$(xy + x) dx + x^2 dy = 0.$$

Solution:

$$M = x(y + 1), \quad N = x^2.$$

$$M_y = x, \quad N_x = 2x.$$

Then

$$\frac{1}{N}(M_y - N_x) = \frac{x - 2x}{x^2} = -\frac{1}{x},$$

which depends only on x . Hence an integrating factor depending on x exists:

$$\mu(x) = \exp\left(\int -\frac{1}{x} dx\right) = \frac{1}{x}.$$

Multiply the equation by $\frac{1}{x}$:

$$(y + 1) dx + x dy = 0.$$

Now

$$\tilde{M} = y + 1, \quad \tilde{N} = x.$$

Check:

$$\tilde{M}_y = 1, \quad \tilde{N}_x = 1,$$

so the equation is exact.

Let $F_x = y + 1$. Then

$$F = x(y + 1) + \phi(y).$$

Differentiate:

$$F_y = x + \phi'(y).$$

Set equal to $N = x$:

$$x + \phi'(y) = x \Rightarrow \phi'(y) = 0.$$

Thus

$$F = x(y + 1).$$

$$\boxed{x(y + 1) = C.}$$

Example 3. Non-exact equation (homogeneous method).

Solve

$$(2xy + y^2) dx + (x^2 + 3xy) dy = 0.$$

Solution:

$$M_y = 2x + 2y, \quad N_x = 2x + 3y \Rightarrow \text{non-exact.}$$

Since homogeneous, let $y = vx$:

$$\frac{dy}{dx} = v + x \frac{dv}{dx}.$$

$$v + x \frac{dv}{dx} = -\frac{2v + v^2}{1 + 3v}.$$

$$x \frac{dv}{dx} = -\frac{v(3 + 4v)}{1 + 3v}.$$

Separate:

$$\frac{1 + 3v}{v(3 + 4v)} dv = -\frac{dx}{x}.$$

Integrate:

$$4 \ln |v| + 5 \ln |3 + 4v| = -12 \ln |x| + C.$$

Back-substitute $v = \frac{y}{x}$:

$$\boxed{x^3 y^4 (3x + 4y)^5 = C.}$$

Lecture 5: Linear First-Order Differential Equations

Linear first-order ODE.

An equation of the form

$$y' + p(x)y = f(x) \quad (1)$$

is called a **linear first-order differential equation**.

Homogeneous case.

If $f(x) \equiv 0$, then (1) becomes

$$y' + p(x)y = 0,$$

and it is called a **homogeneous linear first-order differential equation**. It is separable and has the general solution

$$y = C e^{-\int p(x) dx}.$$

Method: Solving a linear first-order equation using an integrating factor

Consider the linear differential equation

$$y' + p(x)y = f(x).$$

Step 1. Identify $p(x)$ and $f(x)$.

Write the equation in standard form:

$$\frac{dy}{dx} + p(x)y = f(x).$$

Step 2. Compute the integrating factor.

$$\mu(x) = e^{\int p(x) dx}.$$

Step 3. Multiply the equation by $\mu(x)$.

$$\mu(x) \frac{dy}{dx} + \mu(x)p(x)y = \mu(x)f(x).$$

Step 4. Recognize the left-hand side as a derivative.

$$\frac{d}{dx}(\mu(x)y) = \mu(x)f(x).$$

Step 5. Integrate both sides.

$$\mu(x)y = \int \mu(x)f(x) dx + C.$$

Step 6. Solve for y .

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)f(x) dx + C \right).$$

Example 1. Homogeneous linear equation.

Solve

$$\frac{dy}{dx} + 2y = 0.$$

Solution:This is of the form $y' + p(x)y = 0$ with $p(x) = 2$.

$$\mu(x) = e^{\int 2 dx} = e^{2x}.$$

Multiply:

$$e^{2x} \frac{dy}{dx} + 2e^{2x}y = 0.$$

$$\frac{d}{dx}(e^{2x}y) = 0.$$

Integrate:

$$e^{2x}y = C.$$

$$y = Ce^{-2x}.$$

Example 2. Non-homogeneous linear equation.

Solve

$$\frac{dy}{dx} + y = e^x.$$

Solution:

$$\mu(x) = e^{\int 1 dx} = e^x.$$

Multiply:

$$e^x \frac{dy}{dx} + e^x y = e^{2x}.$$

$$\frac{d}{dx}(e^x y) = e^{2x}.$$

Integrate:

$$e^x y = \frac{e^{2x}}{2} + C.$$

$$y = \frac{e^x}{2} + Ce^{-x}.$$

Example 3. Linear equation with variable coefficient.

Solve

$$x \frac{dy}{dx} + y = x^2.$$

Solution:

Divide by x ($x \neq 0$):

$$\frac{dy}{dx} + \frac{1}{x}y = x.$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = x.$$

Multiply:

$$x \frac{dy}{dx} + y = x^2.$$

$$\frac{d}{dx}(xy) = x^2.$$

Integrate:

$$xy = \frac{x^3}{3} + C.$$

$$y = \frac{x^2}{3} + \frac{C}{x}.$$

Lecture 6: Bernoulli Equations

Definition (Bernoulli equation).

An equation of the form

$$y' + p(x)y = f(x)y^\alpha,$$

where $\alpha \neq 0$ and $\alpha \neq 1$, is called a **Bernoulli equation**. It can be reduced to a linear equation by the substitution

$$z = y^{1-\alpha}.$$

Method: Solving a Bernoulli equation

Consider the Bernoulli equation

$$y' + p(x)y = f(x)y^\alpha, \quad (\alpha \neq 0, 1).$$

Step 1. Divide by y^α .

$$y^{-\alpha}y' + p(x)y^{1-\alpha} = f(x).$$

Step 2. Use the substitution

$$z = y^{1-\alpha}.$$

Then

$$\frac{dz}{dx} = (1-\alpha)y^{-\alpha}\frac{dy}{dx}.$$

Step 3. Rewrite the equation in terms of z .

$$\frac{1}{1-\alpha}\frac{dz}{dx} + p(x)z = f(x).$$

Multiply by $1-\alpha$:

$$\frac{dz}{dx} + (1-\alpha)p(x)z = (1-\alpha)f(x).$$

Step 4. Solve the resulting linear equation.

Use integrating factor:

$$\mu(x) = e^{\int (1-\alpha)p(x) dx}.$$

$$\frac{d}{dx}(\mu(x)z) = \mu(x)(1-\alpha)f(x).$$

Step 5. Integrate.

$$\mu(x)z = \int \mu(x)(1-\alpha)f(x) dx + C.$$

Step 6. Substitute back $z = y^{1-\alpha}$.

$$y^{1-\alpha} = \frac{1}{\mu(x)} \left(\int \mu(x)(1-\alpha)f(x) dx + C \right).$$

Example 1.

Solve

$$y' + y = y^2.$$

Solution:

This is a Bernoulli equation with

$$p(x) = 1, \quad f(x) = 1, \quad \alpha = 2.$$

Use the substitution

$$z = y^{1-\alpha} = y^{-1} = \frac{1}{y}.$$

Then

$$z' = -y^{-2}y'.$$

From

$$y' + y = y^2,$$

divide by y^2 :

$$y^{-2}y' + y^{-1} = 1.$$

Since $y^{-2}y' = -z'$ and $y^{-1} = z$, we get

$$-z' + z = 1,$$

or

$$z' - z = -1.$$

This is linear. The integrating factor is

$$\mu(x) = e^{\int -1 dx} = e^{-x}.$$

Multiply:

$$e^{-x}z' - e^{-x}z = -e^{-x}.$$

$$\frac{d}{dx}(e^{-x}z) = -e^{-x}.$$

Integrate:

$$e^{-x}z = e^{-x} + C.$$

Hence

$$z = 1 + Ce^x.$$

Since $z = \frac{1}{y}$,

$$\frac{1}{y} = 1 + Ce^x.$$

Therefore

$$\boxed{y = \frac{1}{1 + Ce^x}}.$$

Example 2.

Solve

$$y' + \frac{1}{x}y = x^2y^3, \quad x \neq 0.$$

Solution:

This is a Bernoulli equation with

$$p(x) = \frac{1}{x}, \quad f(x) = x^2, \quad \alpha = 3.$$

Use the substitution

$$z = y^{1-\alpha} = y^{-2}.$$

Then

$$z' = -2y^{-3}y'.$$

Divide the equation by y^3 :

$$y^{-3}y' + \frac{1}{x}y^{-2} = x^2.$$

Using

$$y^{-3}y' = -\frac{1}{2}z', \quad y^{-2} = z,$$

we obtain

$$-\frac{1}{2}z' + \frac{1}{x}z = x^2.$$

Multiply by -2 :

$$z' - \frac{2}{x}z = -2x^2.$$

This is linear. Its integrating factor is

$$\mu(x) = e^{\int -2/x dx} = x^{-2}.$$

Multiply:

$$x^{-2}z' - \frac{2}{x^3}z = -2.$$

$$\frac{d}{dx}(x^{-2}z) = -2.$$

Integrate:

$$x^{-2}z = -2x + C.$$

So

$$z = x^2(C - 2x).$$

Since $z = y^{-2}$,

$$\frac{1}{y^2} = x^2(C - 2x).$$

Therefore

$$y = \pm \frac{1}{x\sqrt{C - 2x}}.$$

Example 3.

Solve

$$xy' + y = x^2\sqrt{y}, \quad x \neq 0.$$

Solution:

First write the equation in standard form:

$$y' + \frac{1}{x}y = xy^{1/2}.$$

This is a Bernoulli equation with

$$p(x) = \frac{1}{x}, \quad f(x) = x, \quad \alpha = \frac{1}{2}.$$

Use the substitution

$$z = y^{1-\alpha} = y^{1/2} = \sqrt{y}.$$

Then

$$z' = \frac{1}{2}y^{-1/2}y'.$$

Divide the equation by $y^{1/2}$:

$$y^{-1/2}y' + \frac{1}{x}y^{1/2} = x.$$

Since

$$y^{-1/2}y' = 2z', \quad y^{1/2} = z,$$

we get

$$2z' + \frac{1}{x}z = x.$$

Divide by 2:

$$z' + \frac{1}{2x}z = \frac{x}{2}.$$

This is linear. The integrating factor is

$$\mu(x) = e^{\int \frac{1}{2x} dx} = x^{1/2}.$$

Multiply:

$$x^{1/2}z' + \frac{1}{2}x^{-1/2}z = \frac{1}{2}x^{3/2}.$$

$$\frac{d}{dx}(x^{1/2}z) = \frac{1}{2}x^{3/2}.$$

Integrate:

$$x^{1/2}z = \frac{1}{2} \cdot \frac{2}{5}x^{5/2} + C = \frac{1}{5}x^{5/2} + C.$$

Hence

$$z = \frac{1}{5}x^2 + Cx^{-1/2}.$$

Since $z = \sqrt{y}$,

$$\sqrt{y} = \frac{1}{5}x^2 + Cx^{-1/2}.$$

Therefore

$$y = \left(\frac{1}{5}x^2 + Cx^{-1/2} \right)^2.$$