[Solution key]

King Saud University
College of Sciences
Department of Mathematics
Semester 462 / Final Exam / MATH-244 (Linear Algebra)

Max. Marks: 40 **Time: 3 hours**

Solution of Question 1: Correct choices:

(i) If square of a matrix A is zero matrix, then $I - A$ is equal to:

a) 0 b) $(A-I)^{-1}$ c) ✓ $(A+I)^{-1}$ d) $A + I$ [Mark 1]

(ii) If A is a square matrix of order **3** with det(A) = 2, then $det(det(\frac{1}{det(A)}A^3)\,A^{-1})$ is equal to:

a) 1/4 b) ✓ 1/2 c) 1/3 d) 1/16 [Mark 1]

(iii) If the general solution of $AX = O$ is $(-2r, 4r, r), r \in \mathbb{R}$, and $(1,0,-2)$ is a solution of $AX = B$, then the general solution of $AX = B$ is:

a) ✓ $(1-2r, 4r, r-2)$ b) $(-2r, 0, -2r)$ c) $(-2r, 4r, r)$ d) $(-2r-1, 4r, r-2)$ [Mark 1]

(iv) A subset $\boldsymbol{S}$ of $\mathbb{R}^3$ is a basis of the vector space $\mathbb{R}^3$ if $\boldsymbol{S}$ is equal to:

a) ✓ {(1,0,0),(0,2,1),(0,6,0)} b){(1,1,0),(2,1,0),(3,2,0)} c){(1,1,0),(0,0,0),(3,2,1)} d) {(1,1,0),(0,0,1),(2,2,1)} [Mark 1]

(v) If $B = \{u_1 = (2,1), u_2 = (4,3)\}$ and $C = \{v_1 = (0,1), v_2 = (6,0)\}$ are ordered bases of $\mathbb{R}^2$, then the transition matrix $\boldsymbol{P}_{C\to B}$ from C to B is equal to:

a) $\begin{bmatrix} 1 & -1/2 \\ -2 & 3/2 \end{bmatrix}$ b) ✓ $\begin{bmatrix} -2 & 9 \\ 1 & -3 \end{bmatrix}$ c) $\begin{bmatrix} -2/3 & 3 \\ 1/3 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ [Mark 1]

(vi) If B is a square matrix of order 3 with $det(B) = 2$, then $nullity(B)$ is equal to:

a) 2 b) 1 c) 3 d) ✓ 0 [Mark 1]

(vii) If $\langle\,,\rangle$ is an inner product on $\mathbb{R}^n$ and $\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^n$ such that $\|u\|^2 = 5, \|v\|^2 = 1, \langle u,v\rangle = -2$, then $\langle u + 2v, 5u - v\rangle$ is equal to:

a) $\sqrt{5}$ b) ✓ 5 c) 9 d) 41 [Mark 1]

(viii) If $S = \{A, I_2\} \subseteq M_{2\times 2}(\mathbb{R})$, where A is a non-symmetric matrix, then S must be:

a) linearly dependent b) a spanning set for $M_{2\times 2}(\mathbb{R})$ c) ✓linearly independent d) orthogonal [Mark 1]

b) Let $\boldsymbol{T}$ be the transformation from the Euclidean space $\mathbb{R}^2$ to $\mathbb{R}$ given by $\boldsymbol{T}(\boldsymbol{u}) = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^2$, where $\|\boldsymbol{u}\|$ is the Euclidean norm of $\boldsymbol{u}$. Then, for $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ and $k \in \mathbb{R}$, $\boldsymbol{T}$ satisfies:

a) $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})$ b) ✓$T(\boldsymbol{u}+\boldsymbol{v}) \le T(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v})$ c) $T(\boldsymbol{0}) > 0$ d) $T(k\boldsymbol{u}) = kT(\boldsymbol{u})$ [Mark 1]

(x) Zero is an eigenvalue of the matrix $\begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix}$ with geometric multiplicity equal to:

a) 1 b) ✓ 2 c) 3 d) 4 [Mark 1]

Question 2 [Marks $2 + 2 + 3$]:

(a) Find the square matrix A of order **3** such that $A^{-1}(A - I) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and evaluate $det(A)$.

Solution: $\boldsymbol{I} - A^{-1} = A^{-1}(A - I) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow det(A^{-1}) = -1,$ [Mark 1]

and $A = (A^{-1})^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ -2 & -1 & 2 \end{bmatrix}.$ [Mark 1]

(b) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ -2 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & -2 \\ 1 & -1 & -2 \end{bmatrix}$. Find a matrix X that satisfies $XA = B$.

Solution: From Part (a), $A^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix}$; hence, $X = BA^{-1} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 3 & 2 \\ 4 & 1 & 2 \end{bmatrix}$. [Marks 0.5 + 1 + 0.5]

(c) Solve the following system of linear equations:

$$\begin{aligned} x + \quad y + \quad z &= 1 \\ 2x \qquad\quad + 2z &= 3 \\ 3x + 5y + 4z &= 2. \end{aligned}$$

Solution: The matrix of coefficient $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 5 & 4 \end{bmatrix}$ has the inverse $A^{-1} = \frac{1}{-2}\begin{bmatrix} -10 & 1 & 2 \\ -2 & 1 & 0 \\ 10 & -2 & -2 \end{bmatrix}$. [Marks 1.5]

Hence, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$. [Marks 1.5]

[The students may use any one of the methods included in the course MATH-244.]

Question 3 [Marks $3 + 3 + 2$]:

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Then:

(a) Find a basis and the dimension for each of the vector spaces *row*(A), *col*(A), and ***N***(A).

Solution: $RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, $\{(1,0,1), (0,1,1)\}, \{(1,1,2), (0,2,0)\}, \{(1,1,-1)\}$ are bases of [Marks 2]

row(A), *col*(A), ***N***(A), respectively, and so, $dim(row(A)) = 2 = dim(col(A))$, dim(***N***($A$)) = 1. [Mark 1]

(b) Decide with justification whether the following statements are true or false:

(i) $row(A) = row(B)$ (ii) $col(A) = col(B)$ (iii) ***N***(A).= ***N***($\boldsymbol{B}$).

Solution: $RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = RREF\ (B) \Rightarrow row(A) = row(B)$ and ***N***(A) = ***N***($\boldsymbol{B}$). [Marks 1 + 1]

But, $col(A) \neq col(B)$ since $(1,1,2) \notin span(\{(1,0,0), (0,2,0), (1,2,0)\})$. [Mark 1]

(c) Find all square matrices Z of order **3** such that $AZ = O$.

Solution: From Part (a), $\{(1,1,-1)\}$ is a basis of the null space $\boldsymbol{N}(A) = \{X \in \mathbb{R}^3 |\ AX = 0\}$. Hence,

$\boldsymbol{Z} = \begin{bmatrix} a & b & c \\ a & b & c \\ -a & -b & -c \end{bmatrix}$ satisfies $AZ = aA\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + bA\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + cA\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = aO + bO + cO = O$, for all

$a, b, c \in \mathbb{R}$. [Marks 0.5 + 1.5]

Question 4 [Marks $3 + (1 + 3)$]:

(a) Construct an orthonormal basis C of the Euclidean space $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on the given basis $B = \{v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (0,1,1)\}$, and then find the coordinate vector of $v = (1,2,0) \in \mathbb{R}^3$ relative to the orthonormal basis C.

Solution: $u_1 = v_1 = (1,1,0);\ u_2 = v_2 - \frac{<v_2,u_1>}{||u_1||^2}u_1 = (\frac{1}{2}, -\frac{1}{2}, 1);\ u_3 = v_3 - \frac{<v_3,u_1>}{||u_1||^2}u_1 - \frac{<v_3,u_2>}{||u_2||^2}u_2 = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right).$

Hence, $C = \{w_1 = \frac{1}{\sqrt{2}}(1,1,0), w_2 = \frac{1}{\sqrt{6}}(1,-1,2), w_3 = \frac{1}{\sqrt{3}}(-1,1,1)\}$ is the required orthonormal basis of $\mathbb{R}^3$.

Next, $< v, w_1 > = \frac{3}{\sqrt{2}}$, $< v, w_2 > = \frac{-1}{\sqrt{6}}$, and $< v, w_3 > = \frac{1}{\sqrt{3}}$. Hence, $[v]_C = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$. [Marks 1.5 + 0.5 + 1]

(b) Let $\mathscr{P}_2$ denote the vector space of real polynomials with degree ≤ 2. Consider the linear transformation $\boldsymbol{T}: \mathbb{R}^3 \to \mathscr{P}_2$ defined by: $\boldsymbol{T}(1,0,0) = x^2 + 1, \boldsymbol{T}(0,1,0) = 3x^2 + 2, \boldsymbol{T}(0,0,1) = -x^2$. Then:

(i) Compute $\boldsymbol{T}(a, b, c)$, for all $(a, b, c) \in \mathbb{R}^3$.

(ii) Find a basis for each of the vector spaces $Im(\boldsymbol{T})$ and $ker(\boldsymbol{T})$.

Solution: (i) $\boldsymbol{T}(a, b, c) = aT(1,0,0) + bT(0,1,0) + cT(0,0,1) = (a + 3b - c)\, x^2 + \; a + 2b$. [Mark 1]

(ii) From Part (i), $Im(\boldsymbol{T}) = \{(a + 3b - c)\, x^2 + (a + 2b)1 | (a, b, c) \in \mathbb{R}^3\} = span(\{\, x^2, 1\})$, [Mark 1]

and $ker(\boldsymbol{T}) = \{(a, b, c) \in \mathbb{R}^3 | (a + 3b - c)\, x^2 + (a + 2b)1 = 0\}$
$= \{(a, b, c) \in \mathbb{R}^3 |\; a + 3b - c = 0,\; a + 2b = 0\}$
$= \{(a, b, c) \in \mathbb{R}^3 |\; a = -2b, b = c\}$
$= span(\{(-2, 1, 1)\})$. [Mark 1]

Hence, $\{1,\, x^2\}$ and $\{(-2, 1, 1)\}$ are bases of $Im(\boldsymbol{T})$ and $ker(\boldsymbol{T})$, respectively. [Mark 1]

Question 5 [Marks $3 + 2 + 3$]**:** Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix}$. Then:

(a) Find the eigenvalues of A.

Solution: $\boldsymbol{det(A - \lambda I) = det}\left(\begin{bmatrix} 2-\lambda & 2 & -2 \\ 2 & 1-\lambda & -1 \\ 2 & 2 & -2-\lambda \end{bmatrix}\right) = \boldsymbol{\lambda\,(\lambda + 1)(2 - \lambda) = 0}$

$\Rightarrow \lambda = -1, 0, 2$ **are eigenvalues of** A. [Marks 1+1+1]

(b) Find algebraic and geometric multiplicities of all the eigenvalues of A.

Solution: From Part (a), all eigenvalues of A are of same algebraic multiplicity 1. [Mark 0.5]
Next, $E_{-1} = span(\{(2, -1, 2)\}), E_0 = span(\{(0, 1, 1)\})$, and $E_2 = span(\{(1, 1, 1)\})$. Hence, all eigenvalues of A are of same geometric multiplicity 1. [Marks 1.5]

(c) Is the matrix A diagonalizable? If yes, find a matrix P that diagonalizes A.

Solution: Since eigenvalues of A are different, A is diagonalizable. Next, from Part (b), the required matrix:

$$P = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ [Marks 1.5 + 1.5]

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