

Cylindrical Coordinates

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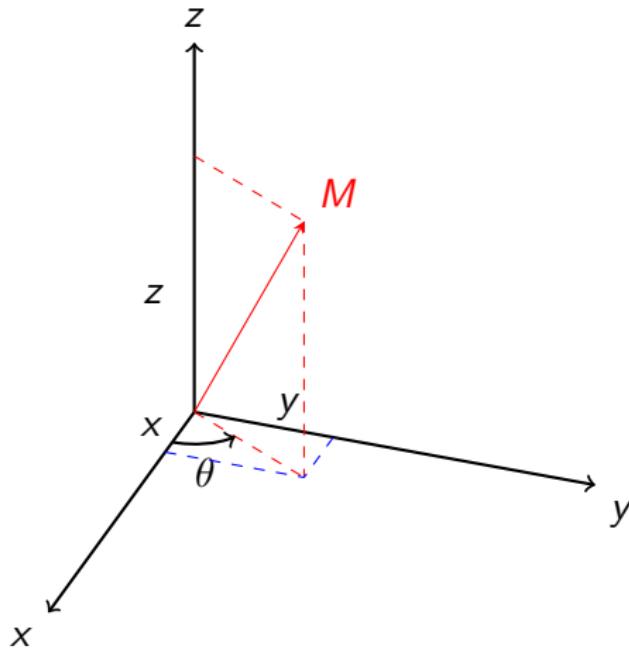
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1 Cylindrical Coordinates

Cylindrical Coordinates

The cylindrical coordinate system is just the polar coordinate system plus the z coordinate. The volume of a typical small unit of volume is $r\Delta r\Delta\theta\Delta z$, or in the limit, $r dr d\theta dz$.



Example

The volume under $z = \sqrt{1 - r^2}$ above the quarter circle inside $x^2 + y^2 = 1$ in the first quadrant.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta &= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1-r^2} \, r \, dr \, d\theta \\ &= \frac{\pi}{6}. \end{aligned}$$

Example

Consider the solid D delimited by both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$, and with density $\rho(x, y, z) = x^2y^2$ at (x, y, z) . Compute its total mass.

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^5 \cos^2 \theta \sin^2 \theta dz dr d\theta &= \frac{1}{2} \int_0^{2\pi} \int_0^1 \sqrt{4-r^2} r^5 \sin^2(2\theta) dr d\theta \\ &= \frac{\pi}{12}. \end{aligned}$$

Double Integrals in Cylindrical Coordinates

Compute the volume under the surface $z = \sqrt{4 - x^2 - y^2}$ and above the quarter of the disc $x^2 + y^2 < 4$ in the first quadrant.

$$V = \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{4 - r^2} r dr d\theta = \frac{4\pi}{3}.$$

Example

Compute the volume under $z = \sqrt{4 - r^2}$ above the disc defined by $0 \leq r < 2 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} V &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} \sqrt{4 - r^2} r dr d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \left(\frac{8}{3} - \frac{8}{3} \sin^3 \theta \right) d\theta = \frac{8\pi}{3} - \frac{32}{9}. \end{aligned}$$

Example

Compute the area outside the circle $r=2$ and inside $r = 4 \sin \theta$.

The region is described by $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$ and $2 \leq r \leq 4 \sin \theta$, so the integral is:

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_2^{4 \sin \theta} r dr d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 8 \sin^2 \theta - 2 d\theta = \frac{4\pi}{3} + 2\sqrt{3}.$$

Exercises

Exercise 1 :

Evaluate $\int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} \frac{(x^2 + y^2)^{\frac{3}{2}}}{x^2 + y^2 + z^2} dz dy dx.$

In cylindrical coordinates, we have:

$$\begin{aligned} \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} \frac{(x^2 + y^2)^{\frac{3}{2}}}{x^2 + y^2 + z^2} dz dy dx &= \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^r \frac{r^3}{r^2 + z^2} dz d\theta dr \\ &= \frac{\pi}{4} \int_0^1 r^2 \left[\tan^{-1}\left(\frac{z}{r}\right) \right]_0^r dr = \frac{\pi^2}{48}. \end{aligned}$$

Exercise 2 :

Evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z \sqrt{x^2 + y^2 + z^2} dz dy dx.$

In cylindrical coordinates, we have:

$$\begin{aligned}
 & \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z \sqrt{x^2 + y^2 + z^2} dz dy dx \\
 = & \int_0^\pi \int_0^1 \int_r^{\sqrt{2-r^2}} r \sqrt{r^2 + z^2} dz dr d\theta \\
 = & \frac{\pi}{3} \int_0^1 2^{\frac{3}{2}} r (1 - r^3) dr = \frac{2^{\frac{3}{2}} \pi}{10}.
 \end{aligned}$$

Exercise 3 :

Evaluate $\int \int \int x^2 dx dy dz$ over the interior of the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 5$.

In cylindrical coordinates, we have:

$$\int \int \int x^2 dx dy dz = \int_0^1 \int_0^{2\pi} \int_0^5 r^3 \cos^2 \theta dz d\theta dr = \frac{5\pi}{4}.$$

Exercise 4 :

Evaluate $\int \int \int xy \, dxdydz$ over the interior of the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 5$.

In cylindrical coordinates, we have:

$$\int \int \int xy \, dxdydz = \int_0^1 \int_0^{2\pi} \int_0^5 r^3 \cos \theta \sin \theta dz d\theta dr = 0.$$

Exercise 5 :

Evaluate $\int \int \int z \, dx dy dz$ over the region above the xy - plane, inside $x^2 + y^2 - 2x = 0$ and under $x^2 + y^2 + z^2 = 4$.

In cylindrical coordinates, we have:

$$\begin{aligned}
 \int \int \int z \, dx dy dz &= \int_0^{2\pi} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z dz r dr d\theta \\
 &= \int_0^{2\pi} \int_0^{2\cos\theta} r(4-r^2) dr d\theta \\
 &= \int_0^{2\pi} \cos^2 \theta d\theta = \pi.
 \end{aligned}$$

Exercise 6 :

Evaluate $\int \int \int yz \, dxdydz$ over the region in the first octant,
 inside $x^2 + y^2 - 2x = 0$ and under $x^2 + y^2 + z^2 = 4$.

In cylindrical coordinates, we have:

$$\begin{aligned}\int \int \int yz \, dxdydz &= \int_0^{2\pi} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} zdzr^2 \sin\theta drd\theta \\ &= \int_0^{2\pi} \int_0^{2\cos\theta} (4-r^2)r^2 \sin\theta drd\theta \\ &= 0.\end{aligned}$$

Exercise 7 :

Evaluate $\int \int \int (x^2 + y^2) dx dy dz$ over the interior of $x^2 + y^2 + z^2 = 4$.

In cylindrical coordinates, we have:

$$\begin{aligned} \int \int \int (x^2 + y^2) dx dy dz &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3 dz dr d\theta \\ &= 4\pi \int_0^2 r^3 \sqrt{4-r^2} dr = \frac{256\pi}{15}. \end{aligned}$$

Exercise 8 :

Evaluate $\int \int \int \sqrt{x^2 + y^2} dx dy dz$ over the interior of $x^2 + y^2 + z^2 = 4$.

In cylindrical coordinates, we have:

$$\begin{aligned}\int \int \int \sqrt{x^2 + y^2} dx dy dz &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 dz dr d\theta \\ &= 4\pi \int_0^2 r^2 \sqrt{4-r^2} dr \\ &\stackrel{r=2\sin t}{=} 16\pi \int_0^{\frac{\pi}{2}} \sin^2 t dt = 4\pi^2.\end{aligned}$$

Exercise 9 :

Compute $\int \int \int (x + y + z) dx dy dz$ over the region inside $x^2 + y^2 + z^2 = 1$ in the first octant.

In cylindrical coordinates, we have:

$$\begin{aligned}
 \int \int \int (x + y + z) dx dy dz &= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} (r \cos \theta + r \sin \theta + z) r dz dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} z r dz dr d\theta \\
 &= 4\pi \int_0^1 r \sqrt{1-r^2} dr = \frac{4\pi}{3}.
 \end{aligned}$$

Exercise 10 :

Compute the mass of a right circular cone of height h and base radius R if the density is proportional to the distance from the base.

In cylindrical coordinates, we have:

$$M = \int_0^R \int_0^{2\pi} \int_0^h z dz r dr d\theta = \frac{1}{2} \pi h^2 R^2.$$

Exercise 11 :

Compute the mass of a right circular cone of height h and base radius R if the density is proportional to the distance from its axis of symmetry.

In cylindrical coordinates, we have:

$$M = \int_0^R \int_0^{2\pi} \int_0^h dz r^2 dr d\theta = \frac{2}{3}\pi h R^3.$$

Exercise 12 :

An object delimited by the unit sphere and has density equal to the distance from the x -axis. Compute the mass.

$$M = \int_0^1 \int_0^{2\pi} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz r^2 |\cos \theta| dr d\theta = 4 \int_0^1 r^2 \sqrt{1-r^2} dr = \frac{\pi}{2}.$$

Exercise 13 :

Compute the volume above the $x - y$ plane, under the surface $r^2 = 2z$, and inside $r = 2$.

Exercise 14 :

Compute the volume inside both $r = 1$ and $r^2 + z^2 = 4$.

Exercise 15 :

Compute the volume below $z = \sqrt{1 - r^2}$ and above the top half of the cone $z = r$.

Exercise 16 :

Compute the volume below $z = r$, above the $x - y$ plane, and inside $r = \cos \theta$.

Exercise 17 :

Compute the volume below $z=r$, above the $x - y$ plane, and inside $r = 1 + \cos \theta$.

Exercise 18 :

Compute the volume between $x^2 + y^2 = z^2$ and $x^2 + y^2 = z$.

Exercise 19 :

Compute the area inside $r = 1 + \sin \theta$ and outside $r = 2 \sin \theta$.

Exercise 20 :

Compute the area inside both $r = 2 \sin \theta$ and $r = 2 \cos \theta$.

Exercise 21 :

Compute the area inside the four-leaf rose $r = \cos(2\theta)$ and outside $r = \frac{1}{2}$.

Exercise 22 :

Compute the area inside the cardioid $r = 2(1 + \cos \theta)$ and outside $r = 2$.

Exercise 23 :

Compute the area of one loop of the three-leaf rose $r = \cos(3\theta)$.

Exercise 24 :

Compute $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(x^2 + y^2) dy dx$ by converting to cylindrical coordinates.

Exercise 25 :

Compute $\int_0^a \int_{-\sqrt{a^2-x^2}}^0 x^2y \, dy \, dx$ by converting to cylindrical coordinates.

Exercise 26 :

Compute the volume under $z = y^2 + x + 2$ above the region $x^2 + y^2 \leq 4$

Exercise 27 :

Compute the volume between $z = x^2y^3$ and $z = 1$ above the region $x^2 + y^2 \leq 1$

Exercise 28 :

Compute the volume inside $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Exercise 29 :

Compute the volume under $z = r$ above $r = 3 + \cos \theta$.

Exercise 30 :

Sketch and describe the cylindrical surface of the given equation.

$$x^2 + z^2 = 1,$$

$$x^2 + y^2 = 9,$$

$$z = \cos\left(\frac{\pi}{2} + x\right),$$

$$z = 9 - y^2,$$

$$z = e^x.$$

Exercise 31 :

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up a triple integral in cylindrical coordinates to find the volume of the region.

The cone is of radius 1. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r$, we have $2 - r^2 = r$, then $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. Thus, the region

$E = \{(r, \theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 \leq z \leq 2 - r^2\}$. Hence the integral for the volume is

$$V = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta = \frac{5\pi}{6}.$$

We can also see that

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1, 0 \leq r \leq z\} \cup \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 1 \leq z \leq 2, 0 \leq r \leq \sqrt{2-z}\}$$

Then

$$V = \int_0^{2\pi} \int_0^1 \int_0^z r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r \, dr \, dz \, d\theta = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$$

Exercise 32 :

Let E be the region bounded below by the xy -plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up a triple integral in cylindrical coordinates to find the volume of the region.

The equation for the sphere is: $x^2 + y^2 + z^2 = 4$ or $r^2 + z^2 = 4$.

The equation for the cylinder is $x^2 + y^2 = 1$ or $r^2 = 1$. Then the region E is defined as follows:

$E = \{(r, \theta, z) | 0 \leq z \leq \sqrt{4 - r^2}, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$. Then the volume of E is

$$\begin{aligned}V &= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta \\&= \int_0^{2\pi} \int_0^1 \left(r \sqrt{4 - r^2} \right) dr \, d\theta \\&= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta = 2\pi \left(\frac{8}{3} - \sqrt{3} \right).\end{aligned}$$

Exercise 33 :

$$\text{Evaluate } \int_0^{\pi/2} \int_0^{\sin \theta} \int_0^{r \cos \theta} r^2 dz dr d\theta.$$

Exercise 34 :

$$\text{Evaluate } \int_0^{\pi} \int_0^{\sin \theta} \int_0^{r \sin \theta} r \cos^2 \theta dz dr d\theta.$$

Exercise 35 :

Compute $\iiint_D f(x, y, z) dxdydz$, for

$$D = \{x^2 + y^2 \leq R^2, 0 \leq z \leq a\},$$

$$f(x, y, z) = x^3 + y^3 + z^3 - 3z(x^2 + y^2),$$

Exercise 36 :

Compute $\iiint_D f(x, y, z) dxdydz$, for

$$D = \{x^2 + y^2 \leq z^2, 0 \leq z \leq 1\}, f(x, y, z) = \frac{z}{(x^2 + y^2 + 1)^2},$$

Evaluate the triple integrals $\iiint_E f(x, y, z) dxdydz$.

- ① $f(x, y, z) = z, E = \{(x, y, z) : x^2 + y^2 \leq 9, x \leq 0, y \leq 0, 0 \leq z \leq 1\}$,

Using the cylindrical coordinates, we get:

$$\iiint_E f(x, y, z) dxdydz = \int_0^3 \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 zdz d\theta dr = \frac{3\pi}{4}.$$

- ② $f(x, y, z) = xz^2$, $E = \{(x, y, z) : x^2 + y^2 \leq 16, x \geq 0, y \leq 0, -1 \leq z \leq 1\}$,

Using the cylindrical coordinates, we get:

$$\iiint_E f(x, y, z) dx dy dz = \int_0^4 \int_{-\frac{\pi}{2}}^0 \int_{-1}^1 r \cos \theta z^2 dz d\theta dr = \frac{8}{3}.$$

- ③ $f(x, y, z) = xy$, $E = \{(x, y, z) : x^2 + y^2 \leq 1, x \geq 0, x \geq y, -1 \leq z \leq 1\}$,

Using the cylindrical coordinates, we get:

$$\iiint_E f(x, y, z) dx dy dz = \int_0^1 \int_0^{\frac{\pi}{4}} \int_{-1}^1 r^2 \cos \theta \sin \theta dz d\theta dr = \frac{1}{3}.$$

- ④ $f(x, y, z) = x^2 + y^2, E = \{(x, y, z) : x^2 + y^2 \leq 4, x \geq 0, x \leq y, 0 \leq z \leq 3\},$

Using the cylindrical coordinates, we get:

$$\iiint_E f(x, y, z) dx dy dz = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^3 r^2 dz d\theta dr = \pi.$$

- ⑤ $f(x, y, z) = e^{\sqrt{x^2+y^2}}, E = \{(x, y, z) : 1 \leq x^2 + y^2 \leq 4, y \leq 0, x \leq y\sqrt{3}, 2 \leq z \leq 3\},$

Using the cylindrical coordinates, we get:

$$\iiint_E f(x, y, z) dx dy dz = \int_1^2 \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \int_2^3 e^r dz d\theta dr = \frac{\pi}{3}(e^2 - 1).$$

- 6) $f(x, y, z) = \sqrt{x^2 + y^2}$, $E = \{(x, y, z) : 1 \leq x^2 + y^2 \leq 9, y \leq 0, 0 \leq z \leq 1\}$, Using the cylindrical coordinates, we get:

$$\iiint_E f(x, y, z) dx dy dz = \int_1^3 \int_{\pi}^{2\pi} \int_0^1 r dz d\theta dr = \pi.$$