CHAPTER 2

SUBGRAPHS-CONNECTED GRAPH

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7 Subgraph

7.1 Definitions

1. **Subgraph:** A *subgraph* of a graph $G = (V(G), E(G))$ is a graph $H = (V(H), E(H))$ verifying:

    - $V(H) \subseteq V(G)$
    - $E(H) \subseteq E(G)$

2. If H is a subgraph of G, we say that G **contains** H (or that H is contained in G, and we write: $G \supseteq H$ (or $H \subseteq G$)).

3. Let $G = (V, E)$ be a graph.

    - A *spanning subgraph* of a graph G is a subgraph H of G such that: $V(H) = V$.
    - For $X \subseteq V$, the subgraph $(X, E \cap [X]^2)$ of G is called the subgraph of G *induced* by X; it's denoted: $G[X]$.

4. Let $G = (V, E)$ be a graph.

    - If $e \in E$, the subgraph $(V, E \setminus \{e\})$ of a graph G is denoted: $G - e$. (Thus $G - e$ is obtained, from G, by deleting the edge e).
    - If $v \in V$, the subgraph $G[V \setminus \{v\}]$ induced by $V \setminus \{v\}$ is denoted by: $G - v$. (Thus, $G - v$ is obtained by deleting from G the vertex v together with all the edges incident with v).

5.
    - A copy of a graph H in a graph G, is a subgraph of G which is isomorphic to H. Such a subgraph is then a H- subgraph of G.
    - For example a K_3-subgraph of G is a triangle of G.

6. An *embedding* of a graph H in a graph G is an isomorphism between H and a subgraph of G ($\exists X \subseteq V, G[X] \simeq H$).

7.
    - A *supergraph* of a graph G is a graph G' which contains G as a subgraph, that is: $(G' \supseteq G)$.
    - Note that each graph G is both a subgraph and supergraph of itself.
    - All other subgraphs H and supergraphs G' are *proper*; we write: $H \subset G$ or $G' \supset G$, respectively.

7.2 Remarks

1. Let $G = (V, E)$ be a graph, $e \in E$, and $v \in V$.

    - $G - e$ is called an edge-deleted subgraph of G.

- $G - v$ is called a vertex-deleted subgraph of G.
- **Note** That any subgraph H of G can be obtained by repeated applications of the basic operations of edge-deletion and vertex-deletion. (for instance, by first deleting the edges of G not in H and then deleting the vertices of G not in H).

2. Given a graph $G(V, E)$, if $e = \{u, v\} \in [V]^2 \setminus E$, the supergraph $(V, E \cup \{e\})$ of G is denoted by: $G + e$.

3. The following theorem due to Erdös (1964/1965), confirms that every graph has an induced subgraph whose minimum degree is relatively large.

Theorem 7.1 *(Erdös)*
Let G be a graph with $d(G) \geq 2k$; where $k \geq 1$ is an integer. Then, G has an induced subgraph H with: $\delta(H) \geq k + 1$.

Proof.
Let G be a graph with $d(G) \geq 2k$; where $k \geq 1$ is an integer, and consider an induced subgraph $H = G[X]$, where $X \subseteq V$ such that:

1. H is with the **largest** possible average degree, that is $d(H)$ is maximum among $d(K)$ where K is induced subgraph of G, $d(K) \leq d(H)$, in particular G is an induced subgraph of G.

2. $|X| = |V(H)|$ is **minimum** among $|V(L)|$ where L is induced subgraph of G with: $d(L) = d(H)$, we notice that: if $|V(K)| < |V(H)|$, then $d(K) < d(H)$.

- **Note** For each graph $K = (V(K), E(K))$, we denoted: $v(K) = |V(K)|$ and $e(K) = |E(K)|$.
- We will show that: $\delta(H) \geq k + 1$.

Fact 1: $v(H) > 1$.
Indeed: If not $v(H) = 1$, then $0 = \delta(H) = d(H)$. But, $d(H) \geq d(G)$, because G is an induced subgraph of G (So by 1. we have $d(H) \geq d(G)$), then $0 \geq d(G) \geq 2k \geq 2$; contradiction.
Fact 2: $\forall x \in V(H), d_H(x) \geq k + 1$.
Indeed: Suppose by contradiction that $\exists x \in V(H) : d_H(x) \leq k$.
Consider the subgraph: $H' = H - x = H[X \setminus \{x\}]$

- **Clearly**, $H' = G[X \setminus \{x\}]$, and then H' is an induced subgraph of G.
- $d(H') = \frac{1}{v(H')} \cdot \sum_{v \in V(H')} d_{H'}(v)$. Then,
  $d(H') = \frac{2e(H')}{v(H')} = \frac{2e(H')}{v(H) - 1}$.
- But, $d_H(x) \leq k$, then: $e(H') \geq e(H) - k$, so, $d(H') = \frac{2e(H')}{v(H) - 1} \geq \frac{2(e(H) - k)}{v(H) - 1}$.
  But, $2k \leq d(G)$, then $d(H') \geq \frac{2e(H) - d(G)}{v(H) - 1} \geq \frac{2e(H) - d(H)}{v(H) - 1} = d(H)$.
  Thus, $d(H') \geq d(H)$; contradiction ($v(H') < v(H)$).

8 Walks-Paths-Cycles

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Definition 8.1
Consider a graph $G = (V, E)$.

1. - *A path P of G from u to v (where $u, v \in V$) is a sequence of vertices $u_0 = u, ..., u_k = v$ such that: $\forall i < k, \{u_i, u_{i+1}\} \in E(G)$, and all the u_i are distinct vertices.*
   - *The length is $l(P)$ the number of edges it uses. (Here, $l(P) = k$).*
   - *P is a uv-path of length k.*

2. **Incidence and Adjacency matrices**
   - *Let $G = (V, E)$ be a graph where: $V = \{v_1, ..., v_n\}$. The adjacency matrix of G is the (n, n) matrix $A_G = (a_{ij})_{1 \leq i,j \leq n}$, where: $a_{ij} = 1$, if $\{v_i, v_j\} \in E$, and $a_{ij} = 0$, if not.*
   - *Let $G = (V, E)$ be a graph where: $V = \{v_1, ..., v_n\}$ and $E = \{e_1, ..., e_m\}$. The incidence matrix of G is the (n, m) matrix $M_G = (m_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$, where: $m_{ij} = 1$, if $v_i \in e_j$, and $m_{ij} = 0$, if not.*

3. *A cycle C of G is a sequence of vertices $u_0, ..., u_k$ forming a u_0u_k-path such that: $\{u_0, u_k\} \in E(G)$ (where $k \geq 2$). We also denote $(u_0, ..., u_k, u_0)$ this cycle.*

4. *Consider a graph $G = (V, E)$.*
   - *Paths in G do not contain repeated vertices or edges.*
   - *Let $u, v \in V$ be a vertices, walk from u to v in G is any sequences of vertices $u = u_0, ..., u_k = v$ such that: $\forall i < k, \{u_i, u_{i+1}\} \in E(G)$.*
   - *A walk in G is any sequences of vertices $u_0, ..., u_k$ such that: $\forall i < k, \{u_i, u_{i+1}\} \in E(G)$. Thus in a walk, edges and vertices may be repeated.*
   - *The length of this walk is the number of its edges (here: k).*
   - *The trail is a walk w where all its edges are distinct.*

Proposition 8.2
Let $u \neq v$ be a two vertices of a graph $G = (V, E)$.
If there is a walk $(u_0 = u, ..., u_k = v)$ from u to v, then we can extract a path from u to v: $u_{i_1} = u, ..., u_{i_p} = v$.

Proof.

- Consider a walk $P = (\alpha_1 = u, ..., \alpha_q = v)$ which is extract from the initial walk $(u_0 = u, ..., u_k = v)$ and which is with **minimum** length (among all extract walks $(\beta_1 = u, ..., \beta_q = v)$). **Note that** the initial walk is extract from itself.

- **Fact** P is a path.
  Indeed: Assume by contradiction that there are $1 \leq i < j \leq p$ such that ; $\alpha_i = \alpha_j$. Thus, $P' = (\alpha_1 = u, ..., \alpha_{i-1}, \alpha_i = \alpha_j, \alpha_{j+1}, ..., \alpha_q = v)$ is an extract walk with: $l(P') < l(P)$. Contradiction.

Proposition 8.3

Let $A_G = (a_{ij})$ be the adjacency matrix of a graph $G = (V, E)$ where $V(G) = \{v_1, ..., v_n\}$. For any integer $k \geq 1$, let $A_G^k = (a_{ij}^{[k]})$. Then for each integer $k \geq 1$, we have: $\forall i, j \in \{1, ..., n\}$; $a_{ij}^{[k]}$ is the number of walks of length k from v_i to v_j.

Proof.

By induction on k.

- For $k = 1$, [there is a walk of length 1 from v_i to v_j] if and only if [$\{v_i, v_j\} \in E(G)$], which case [$a_{ij}^{[1]} = a_{ij} = 1$].
- Assume it's true whenever $1 \leq k \leq t$, and consider A_G^{t+1}. Let $i, j \in \{1, ..., n\}$.
  $a_{ij}^{[t+1]} = \sum_{l=1}^{n} a_{il}^{[t]}.a_{lj} = \sum_{l=1}^{n} N_l$, $(A_G^{t+1} = A_G^t.A_G)$, where: $N_l =$ the number of walks $(\alpha_0 = v_i, ..., \alpha_t = v_l, \alpha_{t+1} = v_j)$ with length $t + 1$ and which terminates by the edge $\{v_l, v_j\}$ (it's deduced from the hypothesis of the induction on $a_{il}^{[t]}$).
- Thus, $a_{ij}^{[t+1]}$ is the number of walks of length $(t + 1)$ from v_i to v_j.

Proposition 8.4

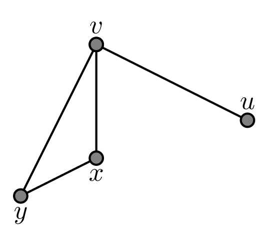
Given a graph $G = (V, E)$, if all vertices of G have degree at least two, then G contains a cycle.

Proof.

Let $P = v_0v_1...v_p$ be a longest path in G. Note that: $p \geq 2$ (because, for $x \in V$ and $y \neq z \in N(x) = \{x, y\}$ we have: yxz is a path in G). As $d(v_p) \geq 2$, there is $v \in N(v_p) \setminus \{v_{p-1}\}$.
If v is not in P (that is: if $v \notin \{v_i; 0 \leq i \leq p\}$), the path $v_0v_1...v_pv$ contradicts the choice of as the longest path.
So, there is i: $0 \leq i \leq p - 2$ such that: $v = v_i$. Thus $v_iv_{i+1}...v_pv_i$ is a cycle in G.

Example 8.5

Consider the graph $G = (\{x, y, u, v\}, \{\{u, v\}, \{v, x\}, \{x, y\}, \{y, v\}\})$



G

The sequence degree of the graph G is $(1, 2, 2, 3)$, where $d_G(u) = 1 < 2$, but the graph G has a cycle $C : xyvx$.

Remarks 8.6

Let $w : v_0 = x, v_1, ..., v_p = y$ an xy-walk.

1. *We say that w connects x to y.*
2. *The vertices x and y are called the ends of the walk, x is its initial vertex and y its terminal vertex.*

3. *The vertices $v_1, ..., v_{p-1}$ are its internal vertices.*

4. *The walk w is closed if $x = y$.*

5. *A cycle of a graph G is closed trail of length ≥ 3, whose initial and internal vertices are distinct.*

9 Connected graphs

Definition 9.1

Let $G = (V, E)$ be a graph, and let u, v be two vertices in V.

1. *Two vertices u and v of G are connected if $u = v$, or if $u \neq v$ and a uv-path exists in G.*

2. *The graph G is connected if $\forall x, y \in V$, x and y are connected.*

3. *The graph G is not connected, we called G is disconnected.*
   Note *that if $|v(G)| \leq 1$, then G is connected.*

Proposition 9.2

Given a graph $G = (V, E)$, for $x, y \in V$ we denote $x\mathcal{C}y$, if x and y are connected. $\mathcal{C}$ is an equivalence relation on the set V.

Proof.

Clearly, $\mathcal{C}$ is reflexive and symmetric.
For the transitivity, consider $u, v, w \in V$, is clear if $u = v$ or $v = w$,
if not assume that: $(u \neq v, u\mathcal{C}v$, and $v \neq w, v\mathcal{C}w)$.
Let $P_1 : u_0 = u, ..., u_p = v$ be a uv-path in G, and $P_2 : v_0 = v, ..., v_q = w$ be a vw-path in G.
Then $W : u_0 = u, ..., u_p = v = v_0, ..., v_q = w$, obtained by concatenating P_1 and P_2, is an uw-walk in G. By Proposition 8.2 , we extract a uw-path in G. Thus, $u\mathcal{C}w$.

Remarks 9.3

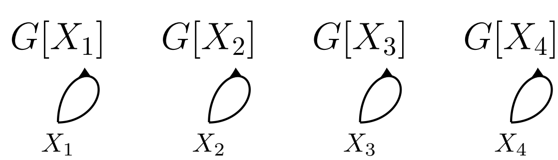
Let $G = (V, E)$ be a graph, and given $\mathcal{C}$ the equivalence relation on the set V.

1. *If X is an equivalence class of $\mathcal{C}$ on V is called connected component, and $G[X]$ is an induced subgraph of the graph G.*

2. *The graph G is connected if $\mathcal{C}$ has at most one class.*

3. *Given a disconnected graph $G = (V, E)$, then for all connected components $X \neq Y$ of G, we have: $\forall (x, y) \in X \times Y$; $\{x, y\} \notin E$. So, the connected components: $X_1, ..., X_k$ of G satisfy:*

   - *The induced subgraphs: $G[X_1], ..., G[X_k]$, are connected.*
   - *the graph G decomposed as:*



4. *A graph $G = (V, E)$ is connected if and only if $\forall x \neq y \in V$, there is an xy-path in G.*

5. *In chapter 1, we consider the following definition:*

   Definition ***9.4***

   - *A graph $G = (V, E)$ is disconnected graph if V can be partitioned into $\{X, Y\}$ such that: ($X \neq \emptyset$, $Y \neq \emptyset$, $\forall (x, y) \in X \times Y : \{x, y\} \notin E$).*
   - *If a graph G is not disconnected, we say that G is connected graph.*
   - *Clearly, a graph $G = (V, E)$ is connected (in this sense) if and only if a graph $G = (V, E)$ is connected (in the sense of the present chapter).*

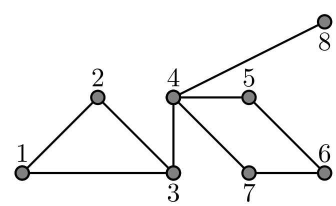
6. *Given a connected component X of graph $G = (V, E)$, we have: $G[X]$ is connected and for each subset Y of V such that: $X \subset Y$, the induced subgraph $G[Y]$ is disconnected.*

7. *If $P := x_0, ..., x_p$ is a path of G, then $\{x_0, ..., x_p\}$ is included in a connected component X of G.*

8. - *If X is a connected component of graph $G = (V, E)$, we can say that the subgraph: $G[X]$ is connected component of G.*
   - *Thus, a connected subgraph H of graph G, is a connected component if and only if H is not contained in any connected subgraph of G having more vertices or edges than H.*

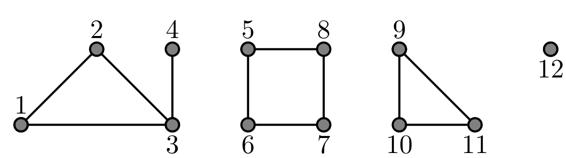
Example 9.5

1. *A graph*
   $G_1 = (\{1, 2, 3, 4, 5, 6, 7, 8\}, \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 4\}, \{4, 8\}\})$
   G_1 *is a connected graph*



G_1

2. *A graph $G_2 = (\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,*
   $\{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{3, 4\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 5\}, \{9, 10\}, \{10, 11\}, \{11, 9\}\})$
   G_2 is a disconnected graph; it has exactly 4 connected components:
   $X_1 = \{1, 2, 3, 4\}$, $X_2 = \{5, 6, 7, 8\}$, $X_3 = \{9, 10, 11\}$ and $X_4 = \{12\}$.



G_2

Notation 9.6 *(***Edge Cuts***)*
Let $G = (V, E)$ be a graph and X, Y be two subsets of V (not necessarily disjoint).

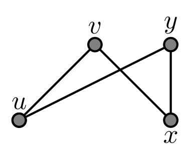
1. *$E[X, Y]$; denotes the set of edges of G with one end in X and the other end in Y.*
2. *$e(X, Y)$ denotes: $|E[X, Y]|$.*
3. *If $Y = X$, the set $E[X, X]$ is denoted $E[X]$ and $e(X, X)$ denoted $e(X)$.*
4. *If $Y = V \setminus X$, the set $E[X, Y] = E[X, V \setminus X]$ is called the* **edge cut** *of G associated with X (or the coboundary of X), and it is denoted by: $\partial(X)$.*
   *(**Note that:** $\partial(X) = E[X, V \setminus X] = \partial(V \setminus X)$).*

Remarks 9.7

1. *If $G = (V, E)$ is a graph, then $\partial(V) = \emptyset$.*
2. *A graph $G = (V, E)$ is bipartite if and only if $\partial(X) = E$ for some proper and non empty subset X of V.*
3. *(A graph $G = (V, E)$ is connected) if and only if ($\forall X \in \mathcal{P}(V) \setminus \{\emptyset, V\}, \partial(X) \neq \emptyset$).*

Example 9.8

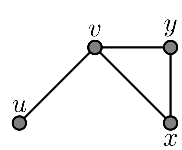
1. *A graph $G_1 = \{x, y, u, v\}, \{\{u, v\}, \{v, x\}, \{x, y\}, \{y, v\}\})$*



G_1

$\partial(\{u, v\}) = \{\{v, x\}, \{y, u\}\}$
$\partial(\{u, x\}) = \{\{u, v\}, \{u, y\}, \{v, x\}, \{x, y\}\}$
$\partial(\{u, v, y\}) = \{\{v, x\}, \{x, y\}\}$.

2. *A graph $G_2 = \{x, y, u, v\}, \{\{u, v\}, \{v, x\}, \{x, y\}, \{y, v\}\})$*



G_2

$\partial(\{u, v\}) = \{\{v, x\}, \{y, v\}\}$
$\partial(\{u, x\}) = \{\{u, v\}, \{v, x\}, \{x, y\}\}$
$\partial(\{u, v, y\}) = \{\{v, x\}, \{x, y\}\}$.

Proposition 9.9

For any graph $G = (V, E)$ and any subset X of V, we have: $|\partial(X)| = \sum_{v \in X} d_G(v) - 2e(X)$.

Proof.

Consider $s = \sum_{v \in X} d_G(v)$. In this sum, each pair $\{x, y\}$ of distinct elements of V, is:

- not counted, if $\{x, y\} \cap X = \emptyset$.
- counted once, if $|\{x, y\} \cap X| = 1$ (that is: if $\{x, y\} \in \partial(X)$).
- counted twice, if $\{x, y\} \subseteq X$ (that is: if $\{x, y\} \in E(X)$).

Thus, $s = 2|E(X)| + 1.|\partial(X)|$.

Theorem 9.10

A graph $G = (V, E)$ is even if and only if $|\partial(X)|$ is even for every subset X of V.
Recall *that G is even if: $d_G(x)$ is even for all $x \in V$.*

Proof.

- " $\Leftarrow$ " Suppose that: $\forall X \subseteq V$, $|\partial(X)|$ is even. So, $\forall v \in V$, $|\partial(\{v\})| = d_G(v)$ is even. Thus, G is even.
- " $\Rightarrow$ " Conversely, if G is even, then, give a subset X of V, we have: $\forall v \in V$, $d_G(v)$ is even and then: $\sum_{v \in X} d_G(v)$ is even, and by Proposition 9.9, $|\partial(X)| = \sum_{v \in X} d_G(v) - 2e(X)$ is even.

Proposition 9.11

Let $G = (V, E)$ be a graph of order $n \geq 1$. G is connected if and only if there is an enumeration: $u_1, ..., u_n$ of its vertices such that: $\forall k \in \{1, ..., n\}$, the induced subgraph $G[\{u_1, ..., u_k\}]$ is connected.

Proof.

- " $\Leftarrow$ " Immediate.
- " $\Rightarrow$ " Let $x \in V$ we will construct $u_1, ..., u_k$ by induction on $k \in \{1, ..., n\}$.
  Let $u_1 = x$. For $k < n$, assume that $u_1, ..., u_k$ are defined such that; $\forall i \leq k$; $G[\{u_1, ..., u_i\}]$ is connected. As $X = \{u_1, ..., u_k\} \in \mathcal{P}(V) \setminus \{\emptyset, V\}$, and G is connected, then: $\partial(X) \neq \emptyset$. So, there is $y \in V \setminus X$ and there is $i \in \{1, ..., k\}$ such that: $\{u_i, y\} \in E$. Thus, we can define u_{k+1} by: $u_{k+1} = y$. Note that: $G[\{u_1, ..., u_k, u_{k+1}\}]$ is connected, (because $\{u_1, ..., u_k, u_{k+1}\} = X \cup \{y\}$, $G[X]$ is connected and y is adjacent to an element of G (at least)).

Remarks 9.12

1. *Given a graph $G = (V, E)$ and a subset X of V such that: $G[X]$ is connected, then: $\forall y \in V \setminus X$, we have: ($G[X \cup \{y\}]$ is connected) if and only if (y is adjacent to at least, an element of X).*
2. *In the proof of Proposition 9.11, we proved that if $G = (V, E)$ is a connected graph, then: for each vertex x of G, there is an enumeration: $u_1 = x, ..., u_n$ of its vertices such that: $\forall k \in \{1, ..., n\}$, the induced subgraph $G[\{u_1, ..., u_k\}]$ is connected.*

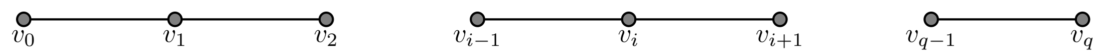
Proposition 9.13

Let $G = (V, E)$ be a connected graph of order $p \geq 2$ such that: $\forall x \in V, d(x) \leq 2$. Then G is a path P_p or a cycle C_p.

Proof.

Let $P = v_0, ..., v_q$ a longest path in G (Note that: $q \geq 1$, and P exists because G is connected).

1. If $V \neq \{v_0, ..., v_q\}$. As G is connected then: $\partial(\{v_0, ..., v_q\}) \neq \emptyset$. So, there is $\alpha \in V \setminus \{v_0, ..., v_q\}$ and there is $0 \leq i \leq q$ such that: $\{\alpha, v_i\} \in E$.
   - If $i = 0$ (resp. $i = q$) then: $P' = \alpha, v_0, ..., v_q$ is a path; contradiction: $(l(P') > l(P))$. (resp. $P' = v_0, ..., v_q, \alpha$ is a path; contradiction: $(l(P') > l(P))$.)
   - If $0 < i < q$; then: $d(v_i) \geq 3$ contradiction.
2. So, $V = \{v_0, ..., v_q\}$.



P_q

As: $\forall i;\ 0 < i < q$: $\{v_{i-1}, v_{i+1}\} \subseteq N_G(v_i)$ and $d(x) \leq 2$ for all $x \in V$, then $\{v_{i-1}, v_{i+1}\} = N_G(v_i)$.
Thus: there are two cases:

- $\{v_0, v_q\} \in E$: then G is a cycle C_q.
- $\{v_0, v_q\} \notin E$: then G is a path P_q.

10 Cut-vertex and Bridges

Notation 10.1

In this section, for each graph $G = (V, E)$, we denoted by $c(G)$, the number of connected components of G.
So, (G is connected) if and only if ($c(G) \leq 1$).

Definition 10.2

Consider a graph $G = (V, E)$, with: $|V| \geq 2$.

1. *For non isolated vertex v, clearly: $c(G - v) \geq c(G)$, we say that v is a cut-vertex if $c(G - v) > c(G)$.*
2. *For each edge e of G, clearly: $c(G - e) \geq c(G)$, we say that e is a bridge if $c(G - e) > c(G)$.*

Remarks 10.3

1. *Let $G = (V, E)$ be a connected graph, u be a vertex of G, and e be an edge of G. Then: (u is a cut-vertex (resp. e is a bridge) of G) if and only if ($G - u$ (resp. $G - e$) is not connected)*
2. *Let $G = (V, E)$ be a graph, v be a non isolated vertex of G, e be an edge of G, X be the connected component of G containing v, and Y be the connected component of G containing e. Then:*
   (a) *(v is a cut-vertex of G) if and only if (v is a cut-vertex of $G[X]$).*
   (b) *(e is a bridge of G) if and only if (e is a bridge of $G[Y]$).*

3. *If v is a cut-vertex of a graph G, then $c(G-v)-c(G)\geq 1$, and we may have:* $c(G-v)-c(G)>1$ *(for example, consider a star).*

Proposition 10.4

Let $G=(V,E)$ be a graph and $e\in E$. If e is a bridge of G, then: $c(G-e)=c(G)+1$.

Remark 10.5

Let $G=(V,E)$ be a graph and $e=\{a,b\}$ be a bridge of G, Y be the connected component of G containing e. $G[Y]-e$ has exactly 2 connected components: X_a and X_b, where $a\in X_a$ and $b\in X_b$.

Proof.

Pose $e=\{a,b\}$, let X_a (resp. X_b) the connected component of $G-e$ containing a (resp. b). Let $t\in Y\setminus\{a,b\}$, where Y is the connected component of G containing e. Consider a ta-path: $P: u_0=t,u_1,...,u_k=a$, of $G[Y]$.

- If ($k\geq 2$ and $u_{k-1}=b$, then: $(u_0=t,u_1,...,u_{k-1}=b)$ is a tb-path of $G-e$, and then: $t\in X_b$.
- If $\forall i 1\leq i\leq k-1,\ u_i\neq b$, then P is ta-path of $G-e$, and then: $t\in X_a$.

It ensues that: $\forall t\in Y\setminus\{a,b\},\ t\in X_a$ or $t\in X_b$.

So, $G[Y]-e$ has at most 2 connected components: X_a and X_b. As, e is a bridge, then $X_a\neq X_b$ and $c(G[Y]-e)=2$. Thus, $c(G-e)=c(G)+1$.

Proposition 10.6

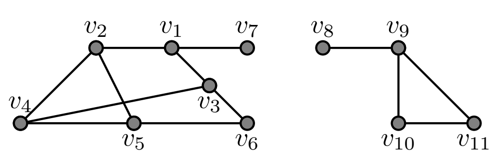
Let $G=(V,E)$ be a graph and e be an edge. Then:
(e is a bridge of G) if and only if (e does not lie an any cycle of G)

Proof.

We may assume that G is connected.
" $\Rightarrow$ " By contraposition, assume that $e=\{u,v\}$ does lie on a cycle $\mathcal{C}:(u,v=u_0,...,u_p=u)$. Then: $G-e$ contains a uv-path; so, in $G-e$; $X_a=X_b$. So, e is not a bridge of G.
" $\Leftarrow$" Conversely, suppose that $e=\{u,v\}$ is an edge which lies on no cycle of G. Assume that, by contradiction, that e is not a bridge of G. Then $G-e$ is connected.
So, there is a uv-path $P:(u_0=u,u_1,...,u_p=v)$ in $G-e$. Thus,: $P+e:(u_0=u,u_1,...,u_p=v,u)$ is a cycle in G; contradiction.

Example 10.7

$G=(\{v_1,v_2,...,v_{11}\},\{\{v_1,v_2\},\{v_1,v_3\},\{v_1,v_7\},\{v_2,v_4\},\{v_2,v_5\},\{v_3,v_4\},$
$\{v_3,v_6\},\{v_4,v_5\},\{v_5,v_6\},\{v_8,v_9\},\{v_9,v_{10}\},\{v_{10},v_{11}\},\{v_{11},v_9\})$



G

- $c(G)=2$
- *The cut-vertices of G are: v_1, v_9 ($c(G-v_1)=3$ and $c(G-v_9)=3$).*
- *The bridges of G are: $\{v_1,v_7\}$, $\{v_8,v_9\}$ ($c(G-\{v_1,v_7\})=3$ and $c(G-\{v_8,v_9\})=3$).*

11 SUBGRAPHS-CONNECTED GRAPHS

Exercise 11.1 *Which pairs of graphs are isomorphic?*
$G_1 = (\{1,2,3,4,5,6,7,8\}, \{1,2\},$
$\{1,5\}, \{1,8\}, \{2,3\}, \{2,6\}, \{3,4\}, \{3,7\}, \{4,5\}, \{4,8\}, \{5,6\}, \{6,7\}, \{7,8\})$.
$G_2 = (\{1,2,3,4,5,6,7,8\},$
$\{1,2\}, \{1,5\}, \{1,8\}, \{2,3\}, \{2,7\}, \{3,4\}, \{3,6\}, \{4,5\}, \{4,8\}, \{5,6\}, \{6,7\}, \{7,8\})$.
$G_3 = (\{1,2,3,4,5,6,7\},$
$\{1,2\}, \{1,4\}, \{1,5\}, \{1,7\}, \{2,3\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,6\}, \{3,7\}, \{4,5\}, \{4,7\}, \{5,6\}, \{6,7\})$.
$G_4 = (\{1,2,3,4,5,6,7\},$
$\{1,2\}, \{1,3\}, \{1,6\}, \{1,7\}, \{2,3\}, \{2,4\}, \{2,7\}, \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{5,6\}, \{5,7\}, \{6,7\})$.

Exercise 11.2

1. (a) *Prove that: if G is a disconnected graph, then the complement graph of G is connected.*
   (b) *Deduce that for all graph G or its complement $\overline{G}$ of G, is a connected graph.*
2. *Show that if $D = (d_n, d_{n-1}, ..., d_2, d_1)$ is graphic, then $\sum_{i=1}^{n} d_i$ is even, and*
$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\}; \qquad \forall k,\ 1 \leq k \leq n.$$
3. *Let $G = (V, E)$ be a graph of order $n \geq 2$. In each of the following cases, show that G is connected*
   (a) $\sum_{v \in V} d(v) > n^2 - 2n$.
   (b) $\forall v \in V,\ d(v) \geq \frac{n-1}{2}$.

Exercise 11.3
Consider a graph $G = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\})$.

1. *Find all walks from v_1 to v_4 of length 3.*
2. *Find all paths from v_1 to v_4 of length 3.*

Exercise 11.4
Prove that:

1. *If G is a nontrivial graph of order n such that $d(u) + d(v) \geq (n-1)$ for every two non adjacent vertices u and v, then G is connected.*
2. *If G is a graph of order n such that $\delta(G) \geq \frac{n-1}{2}$, then G is connected .*