CHAPTER 3

TREES

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12 Definitions- Examples

12.1 Definitions:

1. A graph G is *acyclic or forest* if it contains no cycle.
2. A graph G is a *tree* if G is connected and acyclic.

Remark:
A graph is acyclic if and only if each of its connected components is a tree.

12.2 Examples:

The trees on at most 5 vertices are, up to isomorphy, the following 8 graphs (and the null graph).

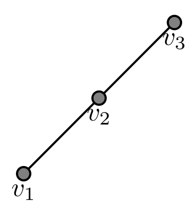
1. Tree with one vertex $T = (\{v\}, \emptyset)$

T

2. Tree with 2 vertices $T = (\{u, v\}, \{\{u, v\}\})$

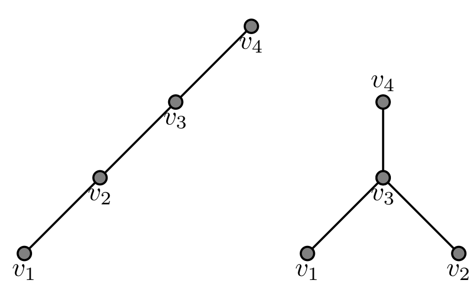
T

3. Tree with 3 vertices $T = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\})$



T

4. Tree with 4 vertices
   $T_1 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\})$,
   and $T_2 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\})$.



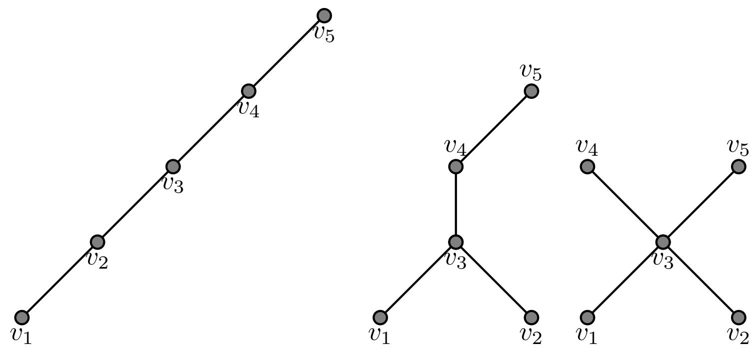
T_1 T_2

5. Tree with 5 vertices

   $T_1 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\})$,

   $T_2 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\})$,
   and

   $T_3 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_3\}, \{v_5, v_3\}\})$.



T_1 T_2 T_2

13 Properties of Trees

13.1 Remarks:

Given a graph $G = (V, E)$ and a set S of edges of G, we can consider the subgraph of G whose edge set in S and whose vertex set is the set of all ends of edges of S. This subgraph is denoted $G[S]$, and called " the edge-induced subgraph $G[S]$".

Note that: an edge-induced subgraphs is simply a subgraph without isolated vertices.

Lemma 13.1
Let $G = (V, E)$ be a graph and $u \neq v$ two vertices.
If G has two distinct paths P_1 and P_2 from u to v, then G contains a cycle.

Proof.
Consider the edge-induced subgraph $H = G[E(P_1) \cup E(P_2)]$. Each vertex of H is of degree ≥ 2, then H contains a cycle (So G also).

Proposition 13.2
In a tree, any two distinct vertices are connected by exactly one path.

Proof.
Let $T = (V, E)$ be a tree and $u \neq v \in V$. As T is connected, then T has at least one path P from u to v. If T has a path $P' \neq P$, from u to v, then by Lemma 13.1, T contains a cycle; contradiction.

Remarks 13.3

1. *As any graph in which all degrees are at least two contains a cycle, then every tree has a vertex of degree at most one.*

2. *Thus, any tree T on $n \geq 2$ vertices has at least a vertex of degree 1. Each such a vertex is called "a leaf of T".*

3. *Given a graph $G = (V, E)$ and a vertex x of G such that: $d_G(x) = 1$. Then:*

   (a) *(G is connected) if and only if ($G - x$ is connected).*

   (b) *(G is acyclic) if and only if ($G - x$ is acyclic).*

4. **Note** *Thus, given a tree $T = (V, E)$ on $n \geq 2$ vertices, then: for each leaf x of T, $T - x$ is a tree.*

Lemma 13.4

Given an acyclic graph $G = (V, E)$ with at least one edge (i.e nonempty acyclic graph), then G has at least two vertices of degree 1.

Proof.

Consider a path $P = (u_1, ..., u_p)$ of G, with maximum length (in particular: $p \geq 2$). Then, $d(u_1) \geq 1$ and $d(u_p) \geq 1$.
If $d(u_1) > 1$ **(resp.** $d(u_p) > 1$**)**, then there is $y \in N(u_1) \setminus \{u_2\}$ (resp. $y \in N(u_p) \setminus \{u_{p-1}\}$).
We distinguish two cases as follows:
Firstly: If $y = u_i$, for some $i \in \{1, ..., p\}$, then $p \geq 3$ and $C = (u_1, u_2, ..., u_i = y, u_1)$ is a cycle of G (resp. $C = (u_i = y, u_{i+1}, ..., u_p, u_i = y)$ is a cycle of G); contradiction.
Secondly: If $y \notin \{u_j; 1 \leq j \leq p\}$, then $P' = (y, u_1, ..., u_p)$ is a path of G and its length l verifies: $l >$ length of P; contradiction.
Thus, $d_G(u_1) = d_G(u_p) = 1$.

Proposition 13.5

Every tree on $n \geq 2$ vertices, has at least two leaves.

Proof.

As a tree T on $n \geq 2$ vertices is an acyclic graph with at least one edges, we conclude by Lemma 13.4

Proposition 13.6

For every tree T on $n \geq 1$ vertices, we have: $e(T) = n - 1$ (i.e: $e(T) = v(V) - 1$).

Proof.

By induction on n. If $n = 1$, then $T \simeq K_1$ and then $e(T) = 0 = v(V) - 1$.
Let $n \geq 1$ be an integer and assume that for every tree T on $n \geq 1$ vertices, $e(T) = n - 1$. Consider a tree $T' = (V, E)$ with: $v(T') = |V| = n + 1$. By Proposition 34.2, we may consider a leaf u of T'. As $T' - u$ is a tree on n vertices, then by hypothesis (of induction); $e(T' - u) = v(T' - u) - 1$. But, $e(T' - u) = e(T') - 1$ (because $d_{T'}(u) = 1$) and $v(T' - u) = |V(T') \setminus \{u\}| = v(T') - 1 = n$. So, $e(T') - 1 = n - 1$. Thus, $e(T') = n = v(T') - 1$.

14 Characterization of Trees

Lemma 14.1

Let $G = (V, E)$ be a graph without isolated vertex such that: $|V| = n \geq 2$ and $|E| = n - 1$. Then G has at least two vertices of degree 1.

Proof.

Let $A = \{u \in V : d(u) = 1\}$ and $B = V \setminus A$. $2|E| = \sum_{u \in V} d(u) = \sum_{u \in A} d(u) + \sum_{u \in B} d(u)$.
So, $2(n-1) = |A| + \sum_{u \in B} d(u)$. As: $\forall u \in B,\ d(u) \geq 2$, then: $2(n-1) \geq |A| + 2|B| = |A| + 2(n - |A|)$.
Thus, $2n - 2 \geq -|A| + 2n$ and then: $|A| \geq 2$.

Theorem 14.2

Let $G = (V, E)$ be a graph with: $|V| = n \geq 1$. The following assertions are equivalent.

1. *G is a tree.*
2. *G is connected and $|E| = n - 1$.*
3. *G is acyclic and $|E| = n - 1$.*

Proof.

- "1. $\Rightarrow$ 2." If G is a tree, then G is connected (by definition) and $|E| = n - 1$ by Proposition 35.2.
- "2. $\Rightarrow$ 3." By induction on n (for "G is acyclic").
  $\rightarrow$ For $n = 1$ or $n = 2$, the result is immediate.
  $\rightarrow$ For $n \geq 2$ such that the result is true for n. Consider a graph $G' = (V', E')$ such that G' is connected, $|V'| = n + 1$ and $|E'| = n$. By Lemma 14.1, consider a vertex u of G' such that: $d_{G'}(u) = 1$ (indeed there are at least two such vertices).
  Clearly, $G' - u$ is connected with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = |E'| - 1 = n - 1$.
  Thus, by hypothesis (of induction), $(G' - u)$ is acyclic. As, $d_{G'}(u) = 1$, then G' is also acyclic.
- "3. $\Rightarrow$ 1." By induction on n.
  $\rightarrow$ For $n = 1$, the result is trivial.
  $\rightarrow$ Assume that the result (i.e. this implication) is true for some $n \geq 1$ and consider a graph $G' = (V', E')$ such that G' is acyclic, $|V'| = n + 1$ and $|E'| = n$. By Lemma 13.4, consider a vertex u of G' such that: $d_{G'}(u) = 1$.
  Thus, $G' - u$ is (also) acyclic with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = n - 1$.
  Thus, by hypothesis (of induction), $(G' - u)$ is a tree, then is connected. Thus, G' is also connected (and then G' is a tree, because G' is acyclic by hypothesis).

15 Spanning trees

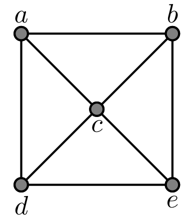
Definition 15.1

1. *Given a graph $G = (V, E)$, a subgraph of G which is a tree is called "a subtree of G".*

2. *A spanning subgraph of G which is a tree is called spanning tree of G.*

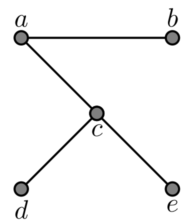
Example 15.2

Let $G = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{e, d\}, \{e, c\}, \{d, c\}\})$ be a graph.



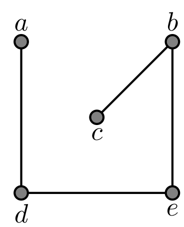
G

Then $T_1 = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{e, c\}, \{d, c\}\})$ is spanning tree of G.



T_1

Then $T_2 = (\{a, b, c, d, e\}, \{\{a, d\}, \{e, d\}, \{b, e\}, \{b, c\}\})$ is spanning tree of G.



T_2

Remarks 15.3

1. *Recall that a result in chapter 2: Given an edge e of a graph G,*
   (e is a bridge of G) if and only if (e does not lie on any cycle of G).

2. *A bridge of G which is also called " a cut edge of G" (See Book of "Bondy and Murty").*

Theorem 15.4

A graph is connected if and only if it has a spanning tree.

Proof.

- "$\Leftarrow$" If a graph G has a spanning tree T, then two distinct vertices of G (and then of T) are connected by a path in T, which is also a path in G; so G is connected.

- "$\Rightarrow$" Consider a connected graph $G = (V, E)$ which is not tree. Given an edge e of a cycle of G, $G - e = (V, E \setminus \{e\})$ is a spanning subgraph of G which is also connected (because by remark e is not a bridge of G). By repeating this process of deleting edges in cycles until every edge which remains does not lie in a cycle and then it is a cut edge (i.e bridge), we obtain a spanning subgraph T of G such that: T is connected and acyclic. So, T is a spanning tree of G.

Corollary 15.5

If $G = (V, E)$ is a connected graph with $|V| = n \geq 1$, then $|E| \geq n - 1$.

Proof.

By Theorem 28.3, G has a spanning tree T. By Theorem 28.1, $|E(T)| = n - 1$. As, $E(T) \subset E(G) = E$, then: $|E| \geq n - 1$.

16 Characterization of bipartite Graphs

Remarks 16.1

1. *A graph G is a bipartite if and only if each of its connected components is bipartite.*
2. *A graph G contains an odd cycle if and only if one of its connected components contains an odd cycle.*

Theorem 16.2

A graph is bipartite if and only if it contains no odd cycles.

Proof.

By remark 16.1, it suffices to prove the theorem in the case of connected graphs.

- "$\Rightarrow$" Let $G = G[X, Y]$ be a connected bipartite graph. The vertices of any path in G belong alternately to X and to Y. Thus, all paths connecting 2 vertices in different parts (one in X and the other in Y) is of odd length, and all paths connecting 2 vertices in the same part (X or Y) are of even length.
  It follows that every cycle $(\alpha_1, \alpha_2, ..., \alpha_p, \alpha_1)$ in G is of even length (because α_1 and α_p are in different parts, so $l(\alpha_1, \alpha_2, ..., \alpha_p)$ is odd and then $l(\alpha_1, \alpha_2, ..., \alpha_p, \alpha_1)$ is even).

- "$\Leftarrow$" Conversely; consider a connected graph G without odd cycles: By Theorem 28.3, G has a spanning tree $T = (V, E_1)$ (where $G = (V, E)$). We may assume that: $|V| \geq 2$.
  Let $x \in V$. By Proposition 30.2, for each $v \in V \setminus \{x\}$, there is only one path P_v, in T, connecting v to x.
  Let: $X = \{x\} \cup \{v \in V \setminus \{x\} : l(P_v) \text{ is even}\}$ and $Y = V \setminus X$.
  Note That: $Y = \{v \in V \setminus \{x\} : l(P_v) \text{ is odd}\}$, and $Y \neq \emptyset$ (because $N_T(x) \neq \emptyset$ and $N_T(x) \subseteq Y$).

  Thus, $\{X, Y\}$ is a partition of V (because $X \neq \emptyset$, $Y \neq \emptyset$, and $X \cap Y = \emptyset$).

  Fact 1: $T[X]$ is an empty graph.

  Indeed:
  $\rightarrow \forall \alpha \in X,\ \{\alpha, x\} \notin E(T)$ because $N_T(x) \subseteq Y$.

$\rightarrow$ Assume that $T[X]$ is a nonempty graph. So, there is an edge $\{\alpha, \beta\} \subseteq X \setminus \{x\}$.
Assume, for example, that $l(P_\alpha) \leq l(P_\beta)$ and let $P_\alpha = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha)$. As, $l(P_\alpha) \leq l(P_\beta)$, then $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So, $P'_\alpha = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha, \beta)$ is a path in T connecting x to β. It ensues, that: $l(P_\beta) = l(P'_\alpha) = l(P_\alpha) + 1$. Then $l(P_\beta)$ is odd (because $l(P_\alpha)$ is even; $(\alpha \in X)$); contradicts the fact that $\beta \in X$.

Fact 2: $T[Y]$ is an empty graph.

Indeed:
Assume that $T([Y]$ is a nonempty graph and consider an edge $\{\alpha, \beta\}$ of $T([Y]$ with: $l(P_\alpha) \leq l(P_\beta)$. Denote: $P_\alpha = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha)$. Then, $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So, $P_\beta = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha, \beta)$ is a path in T connecting x to β. So, $l(P_\beta) = l(P_\alpha) + 1$. Contradiction.

$\rightarrow$ Thus, T is bipartite graph with the partition $\{X, Y\}$.

Now, we will prove that G is bipartite with $\{X, Y\}$ as a bipartition.
For this, consider an edge $e \in E \setminus E_1 = E(G) \setminus E(T)$. Denote: $e = \{u, v\}$.
Let $P = (\alpha_1 = u, \alpha_2, ..., \alpha_q = v)$ the unique path in T, from u to v. By hypothesis, the cycle $C = P + e$ is of even length. Thus, $l(P) = l(C) - 1$ is odd. As P is a path of the bipartite graph $T = T[X, Y]$, then u and v belong to distinct parts (i.e one is in X and the other is in Y).
Thus, every edge e' of G is such that: one end of e' is in X and the other end is in Y.
So G is bipartite and $G = G[X, Y]$.

Corollary 16.3

Every tree is a bipartite graph.

Proof.

As a tree is without cycles, this corollary is an immediate consequence of Theorem 28.5.

17 Exercises of Trees

N.B: All the graphs considered here are nontrivial

Exercise 17.1

Let F be a forest of order p and size q having k connected components. Obtain an expression for q in terms of p and k.

Exercise 17.2

1. (a) *Find all the trees, up to isomorphism, T is a regular graph.*
   (b) *Find all the trees, up to isomorphism, T is a complete bipartite graph $K_{p,q}$ (where $p, q \geq 1$).*
2. (a) *Find all the trees T, up to isomorphism, on $\{1, 2, ..., n\}$ where $n \geq 3$, and T has exactly $n - 1$ leaves.*
   (b) *Find all the trees T, up to isomorphism, on $\{1, 2, ..., n\}$ where $n \geq 3$, and T has exactly 2 leaves.*

Exercise 17.3

Given an integer $n \geq 2$, show that an increasing sequence $D = (d_1, d_2, ..., d_n)$ of positive integers is the degree sequence of some tree if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

(Hint: Prove that if $n \geq 3$, and $\sum_{i=1}^{n} d_i = 2n - 2$, then $d_1 = 1$ and $d_n > 1$, and use an induction on n.)

Exercise 17.4

Show that any tree T has at least $\Delta(T)$ leaves.

Exercise 17.5

Let T be a tree with $v(T) = n \geq 3$, and denote $x_i = |\{v \in V(T) : d_T(v) = i\}|$.

1. *Show that* $\sum_{i=3}^{n-1} (i - 2)x_i = x_1 - 2,$
2. *Count the number of nonisomorphic trees that have 5 leaves and no vertices of degree 2.*

Exercise 17.6

Let G be a connected graph.

1. *Let T be a spanning tree of G. Show that every cycle of G has an edge that is in the complement $\overline{T}$.*
2. *Let $e \in E(G)$. Show that e is a bridge of G if and only if e belongs to every spanning tree of G.*

Exercise 17.7

Let T be a tree of order k. Show that any graph G with $\delta(G) \geq k-1$ has a subgraph isomorphic to T.

(Hint: Use an induction on k).

Exercise 17.8

Let T be a tree of order $n \geq 2$.

1. *Let $u \in V(T)$, and suppose that $P : (v_1 = u, v_2, ..., v_p = v)$ is a maximal path in T such that u is an end of P. Show that v is a leaf of T.*
2. *Show that there is a vertex v of T such that $N_T(v)$ consists of leaves excepting possibly one neighbour.*

Exercise 17.9

Let $A \subseteq \mathbb{N}$ such that $0 \notin A$ and $1 \in A$. Show that there is a tree T such that $A = \{d_T(v) : v \in V(T)\}$.
(Hint: Use an induction on $n = |A|$).

Exercise 17.10

1. *Define the Prüfer code $b(T) = (p_1, p_2, ..., p_{n-2})$ of a tree T of order n.*
2. *Prove Cayley's Theorem: There are exactly n^{n-2} trees on $\{1, 2, ..., n\}$.*