Introduction to Discrete Structures

November 20, 2024

We will be only studying simple and finite graphs during this course

CHAPTER 1

GRAPHS

1 Definitions and Examples

1.1 Definitions

1. **Graph :** A *graph* is an ordered pair $G = (V(G), E(G))$ (or simply $G = (V, E)$) where, $V(G)$ is a finite set, and $E(G)$ is a subset of $[V]^2$ ($[V]^2$ is the set of the pairs $\{u, v\}$ such that $u \neq v$).

2. Let $G = (V, E)$ be a graph
   (a) Each element of V is a *vertex* of G.
   (b) V is the *vertex set* of G.
   (c) Each element of E is an *edge* of G.
   (d) E is the *edge set* of G.

3. Occasionally, it is desirable to denote $V(G)$ the vertex set of a graph G and $E(G)$ its edge set. This is useful when we have two or more graphs under consideration.

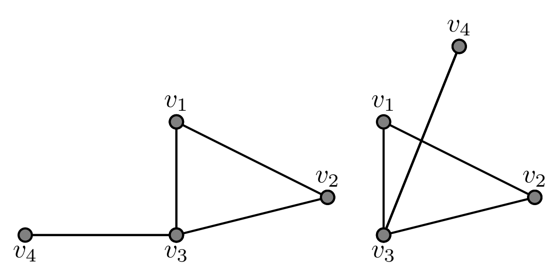
4. Let $G = (V, E)$ be a graph
   (a) The *order* of G denoted by: $|G|$ is the number $|V|$.
   (b) The *size* of G denoted by: $\|G\|$ is the number $|E|$.

5. Let $G = (V, E)$ be a graph. An edge $\{u, v\}$ is denoted simply uv.

6. It is convenient to represent a graph by a diagram.
   In such representation, we indicate the vertices by points (or small circles), and we represent the edges by line segments (or curves) joining the two appropriate points.

1.2 Examples

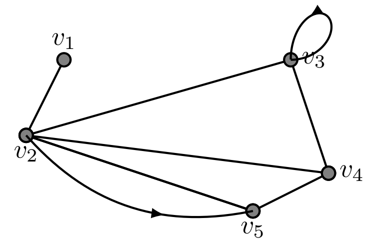
1. Let $G = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_1v_3, v_2v_3, v_3v_4\})$ be a graph



Two representations of the same graph G

   Order of G is 4
   Size of G is 4

2. Let $H = (\{v_1, v_2, v_3, v_4, v_5\}, \{v_1v_2, v_2v_3, v_3v_3, v_3v_4, v_2v_4, v_4v_5, v_2v_5, v_2v_5\})$ be a graph



Representation of H, the graph H is not a simple graph

Order of H is 5
Size of H is 8
We remak that, in this case, the graph H is not simple, because H has a double (multiple) edges (or because H has a loop).

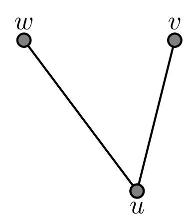
1.3 Definitions

1. Let $G = (V, E)$ be a graph.

   (a) For $x \neq y \in V$, we say that the vertices x and y are *adjacent* when $\{x, y\}$ is an edge. If not, the vertices x and y are *nonadjacent*.

   (b) If $e = \{x, y\}$ is an edge, x and y are the *ends* of e and x (and y) is *incident* with (to) the edge e.

   (c) If uv and uw are different edges (i.e: $v \neq w$) we say that the edges uv and uw are *adjacent*.
   Let $G = (\{u, v, w\}, \{uv, uw\})$ be a graph



Representation of G with the edges uv and uw are adjacent.

2. Let $G = (V, E)$ be a graph, and let v be a vertex of G.

   (a) Two adjacent vertices are *neighbours*.

   (b) The set of neighbours of vertex v, called the *neighborhood* of v; is denoted by: $N_G(v)$ (or simply $N(v)$).
   Let S be a subset of V. The *neighborhood of S*, denoted by $N(S)$, is the set of vertices in V that have an adjacent vertex in S. The elements of $N(S)$ are called the *neighbours* of S, noted that: $N(\{v\}) = N(v)$.

   (c) The *degree* of the vertex v is the number $\mid N_G(v) \mid$ denoted by: $d_G(v)$ or $deg(v)$ (or simply $d(v)$).
   A vertex v of the graph G is called *vertex even* or *vertex odd* according to the parity of

its degree.
A vertex v of the graph G is called *isolated vertex*, if $d_G(v) = 0$, and a vertex of degree 1 in G is called a *leaf*.

(d) The *maximum degree of the vertices* of G is denoted: $\Delta(G)$.

(e) The *minimum degree of the vertices* of G is denoted: $\delta(G)$.

(f) The *average degree of the vertices* of G denoted by: $d(G)$ such that, $d(G) = \dfrac{1}{n}\displaystyle\sum_{v \in V} d(v)$, where $n = |V| \geq 1$.

N. B: It is easily to see that: $\delta(G) \leq d(G) \leq \Delta(G)$.

3. Given a graph G, with the vertex set $V = \{v_1, v_2, ..., v_n\}$, the sequence $(d(v_1), ..., d(v_n))$ is called the *degree sequence* of G.

1.4 Remarks

1. Given a graph $G = (V, E)$, we denote $v(G) = |V|$ and $e(G) = |E|$.
2. In the book of Bondy and Murty:
   - The term "graph" always means 'finite graph', we call a graph with just one vertex *trivial* and all other graphs *nontrivial*.
   - Much of graph theory is concerned with the study of simple graphs.
   - The graph with no vertices (and then no edges) is the *null graph*.
     Unless otherwise specified, we consider *non null graphs* (i.e: $V(G) \neq \emptyset$).
3.
   - Given a graph of order n, we can enumerate his vertices by: $v_1, v_2, ..., v_n$ such that, $d(v_1) \leq ... \leq d(v_n)$.
   - The increasing (or decreasing) sequence $(d(v_1), ..., d(v_n))$ is the *degree sequence* of G.

2 Vertex degrees

2.1 Properties of vertex degrees

Proposition 2.1 *Let G be a graph, $\delta(G) \leq d(G) \leq \Delta(G)$.*

Theorem 2.2 ***(Handshake lemma)***
For any graph G, the sum of the degrees of the vertices of G equals twice the number of edges of G. (i.e: $\displaystyle\sum_{v \in V} d(v) = 2|E|$, where $G = (V, E)$).

Proof.
Let $G = (V, E)$ be a graph and consider the sum $S = \displaystyle\sum_{v \in V} d(v)$. For $a \neq b \in V$, we count the edge $\{a, b\}$ **twice** if $\{a, b\} \in E$ (one in $d(a)$ and one in $d(b)$), and we don't count the edge $\{a, b\}$ if $\{a, b\} \notin E$. So, $S = 2|E|$.

Corollary 2.3

Every graph contains an even number of odd vertices.

Proof.

Let $G = (V, E)$ be a graph and consider $V(G) = A \cup B$ where, A (resp. B) is the set of even (resp. odd) vertices of G.

We have $\sum\limits_{v\in V} d(v) = \sum\limits_{v\in A} d(v) + \sum\limits_{v\in B} d(v) = 2|E|$, hence $\sum\limits_{v\in B} d(v) = 2|E| - \sum\limits_{v\in A} d(v)$, then $\sum\limits_{v\in B} d(v)$ is even. It ensues that $|B|$ is even. (**Note:** $\sum\limits_{v\in\emptyset} d(v) = 0$).

Theorem 2.4 *(***Pigeonhole Principle***)*

Let S be a finite set with $|S| = n$, and let $S_1, ..., S_k$ be a partition of S into k subsets such that: $1 \leq k < n$. Then at least one subset S_i contains at least $(\lfloor \frac{n-1}{k} \rfloor + 1)$ elements (let y be a real number, $\lfloor y \rfloor$ is the greatest integer p, $p \leq y$, and $\lfloor y \rfloor$ is called the floor of y).

Proof.

By contradiction. If not: $\forall i \in \{1, ..., k\}$, $|S_i| \leq \lfloor \frac{n-1}{k} \rfloor$.

So, $|V| = \sum\limits_{1\leq i\leq k} |S_i| \leq k.\dfrac{n-1}{k} = n - 1 < n$.

Thus $|V| = n < n$; contradiction.

Corollary 2.5

Given a graph $G = (V, E)$ on $n \geq 2$ vertices, there are $x \neq y \in V$ such that: $d(x) = d(y)$.

Proof.

Given $G = (V, E)$ a graph, the first remark, if there is an isolated vertex x (i.e: $d(x) = 0$), then: $(\forall y \in V,\ d(y) \leq n - 2)$ and the second remark, if there is a vertex x such that $d(x) = n - 1$, then: $(\forall y \in V,\ d(y) \geq 1)$.

By the first remark and the second remark we deduce $(\forall v \in V, d(v) \in \{0, ..., n - 2\})$ or $(\forall v \in V,\ d(v) \in \{1, ..., n - 1\})$.

Thus, the n values: $d(v_1), ..., d(v_n)$ (where $V = \{v_1, ..., v_n\}$) are all in set A with: $|A| = n - 1$. So, we conclude by the **Pigeonhole Principle**.

Corollary 2.6 *(***Particular case of Pigeonhole Principle***)*

If we put n pigeons in k cages such that $k < n$, then at least one cage contains at least two pigeons.

2.2 Exercises

1. (a) Show that there is no graph with degree sequence: $(2, 3, 3, 4, 4, 5)$.

   (b) Show that there is no graph with degree sequence: $(2, 3, 4, 4, 4, 6, 6, 6, 9)$.

   (c) Show that there is no graph with degree sequence: $(1, 3, 3, 3)$.

   (d) Show that there is no graph with degree sequence: $(1, 2, 4, 5, 6, 6, 7, 8, 9)$.

   (e) Show that there is no graph with degree sequence: $(1, 2, 3, 4, 4)$.

   (f) Show that there is no graph with degree sequence: $(2, 3, 4, 5, 5, 5)$.

2. Show that, given a group of $n \geq 2$ students, there are at least two students (from this group) having the same number of friends (in the group).

3. We have 15 computers. Is it possible to connect each of them to exactly 3 others?

4. Let p, n two odd integers, such that $p < n$. We have n computers. Is it possible to connect each of them to exactly p others?

3 Isomorphic Graph

3.1 Definitions

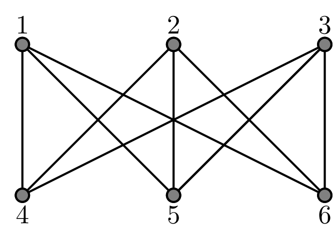
1.
   - An *isomorphism* from graph $G = (V(G), E(G))$ onto a graph $H = (V(H), E(H))$ is a bijection $f : V(G) \to V(H)$ such that: $\forall x, y \in V(G), (xy \in E(G) \Leftrightarrow f(x)f(y) \in E(H))$.
   - We say that G is *isomorphic* to H (or G and H are isomorphic), and we denoted $G \simeq H$ (or $G \approx H$ or $G \cong H$), if there exists an isomorphism from G onto H.

2.
   - An isomorphism from a graph G onto G itself is called: an *automorphism of* G.
   - The set of automorphisms of G is denoted: $Aut(G)$.

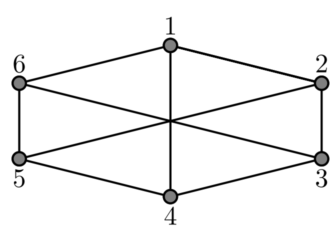
3.
   - The *complement of a graph* $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ where, $\overline{E} = [V]^2 \setminus E$ (So, $\forall x \neq y \in V,\ (xy \in \overline{E} \Leftrightarrow xy \notin E)$).
   - A graph G is called *self-complementary* if it is isomorphic to its complement $\overline{G}$.

3.2 Examples

1. Let $G_1 = (\{1, 2, 3, 4, 5, 6\}, \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\})$ and let $G_2 = (\{1, 2, 3, 4, 5, 6\}, \{\{1, 2\}, \{1, 4\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 6\}, \{4, 5\}\{5, 6\}\})$

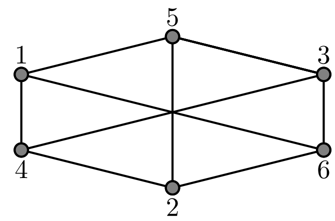


G_1



G_2

The drawing of G_1 can be transformed into the following G_2 by first moving vertex 2 to the bottom of the diagram, and the moving 5 to the top, we obtained the diagram of the graph G_1 as follows:



G_1

So, $f = \binom{1\,2\,3\,4\,5\,6}{6\,4\,2\,5\,1\,3}$ is an isomorphism from G_1 onto G_2.

2. Let $G_3 = (\{x, y, z, u, v, w\}, \{xy, xz, yz, uv, uw, vw, xu, yv, zw\})$

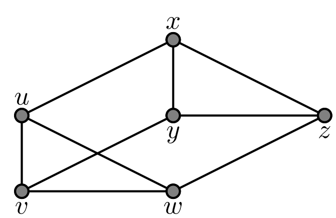
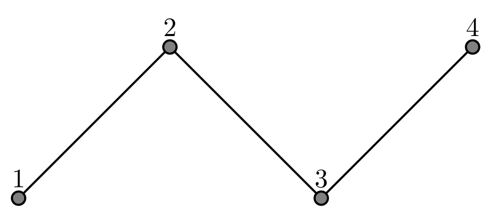
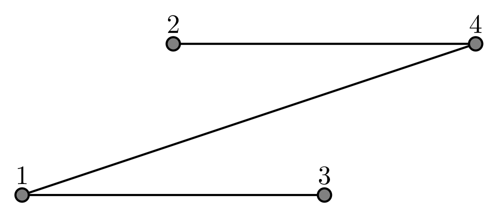


Diagram of G_3 is *the graph of prism.*
G_3 is not isomorphic to G_2 ($G_3 \ncong G_2$).

3. • Let $H_1 = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}\})$ be a graph,
$\overline{H_1} = (\{1, 2, 3, 4\}, \{\{1, 3\}, \{1, 4\}, \{2, 4\}\})$

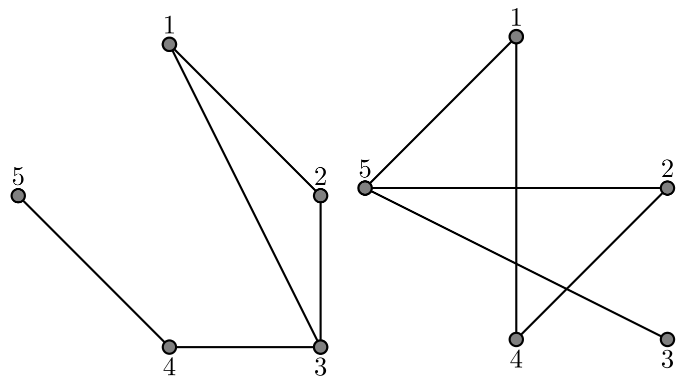




H_1 $\overline{H_1}$

The graph H_1 is self-complementary graph.

• Let $H_2 = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ be a graph,
$\overline{H_2} = (\{1, 2, 3, 4, 5\}, \{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$



H_2 $\overline{H_2}$

The graph H_2 is not self-complementary graph.

3.3 Remarks

1. The relation " is isomorphic to" is an equivalence relation on the class of all graphs.

   Indeed:

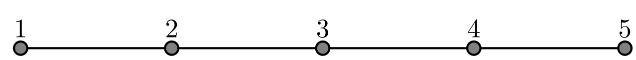
   - Let $G = (V, E)$ be a graph, we have id_V is an isomorphism from G onto G.
   - Let G and G' be two graphs, if f is an isomorphism from G onto G', the inverse mapping f^{-1} is an isomorphism from G' onto G.
   - The composite mapping, $f_2 \circ f_1$, of two isomorphisms is an isomorphism.

2. Given an isomorphism f from $G = (V, E)$ onto $G' = (V', E')$, then:

   - $\forall x \in V,\ f(N_G(x)) = N_{G'}f(x)$; so $d_{G'}(f(x)) = d_G(x)$.
   - $|V| = |V'|$, $|E| = |E'|$, and G and G' have the same degree sequence.

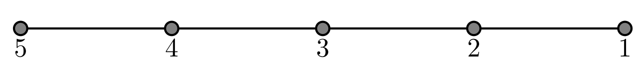
3. Let $G(V, E)$ be a graph. $(Aut(G), \circ)$ is a group (it is a subgroup of $(S_V, \circ)$ the group of permutations of V). The group $(Aut(G)$ is called the *automorphism group of* G.

   Example
   Let $G = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ be a graph.



G

The graph G is too $G = (\{1, 2, 3, 4, 5\}, \{\{5, 4\}, \{4, 3\}, \{3, 2\}, \{2, 1\}\}$.



G

$f = \binom{1\ 2\ 3\ 4\ 5}{5\ 4\ 3\ 2\ 1}$ is an automorphism.

3.4 Properties

Proposition 3.1

If a graph G is self-complementary, then his order n satisfies: $n \equiv 0 \ (mod\ 4)$ or $n \equiv 1 \ (mod\ 4)$ (i. e $n = 4p$ or $n = 4p + 1$, where $p \in \mathbb{N}$).

Proof.

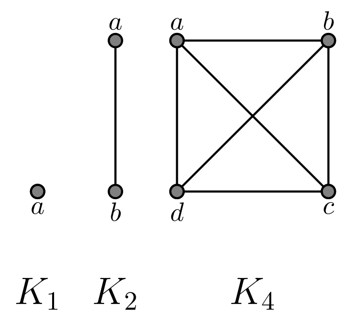
Let $G = (V, E)$ be a self-complementary graph, then $|E| = |[V]^2 \setminus E|$, hence $2|E| = |[V]^2| = \binom{n}{2} = \frac{n(n-1)}{2}$, therefore $|E| = \frac{n(n-1)}{4}$, since $|E| \in \mathbb{N}$, hence 4 devises $n(n-1)$, therefore $n \equiv 0 \ (mod\ 4)$ or $n \equiv 1 \ (mod\ 4)$.

4 Particular Graphs

4.1 Complete Graph

- A *complete graph* is a graph in which any two vertices (different vertices) are adjacent.
- Up to isomorphy, for each integer $n \geq 1$, there is a unique complete graph of order n. It is denoted: K_n.

Examples:



4.2 Empty Graph

- An *empty graph* is a graph $G = (V, E)$ with: $E = \emptyset$.
- Up to isomorphy, for each integer $n \geq 1$, there is a unique empty graph of order n. It is denoted: D_n.

Examples:

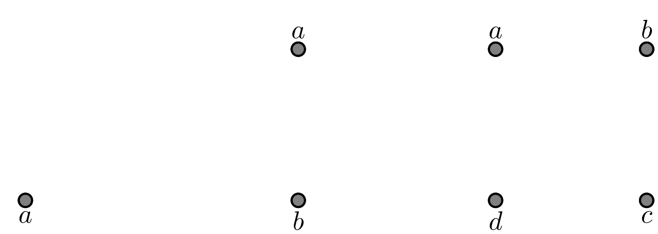


Diagram of D_1 Diagram of D_2 Diagram of D_4

4.3 Paths

A *path* is a graph isomorphic to the graph: $P_n = (\{1, ..., n\}, \{\{i, i+1\};\ 1 \leq i \leq n-1\})$.

Examples:

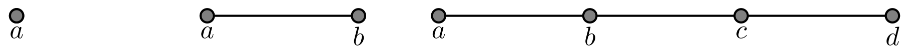


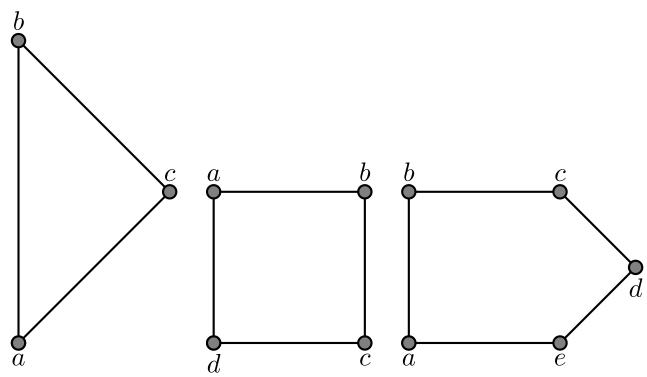
Diagram of P_1 Diagram of P_2 Diagram of P_4

4.4 Cycles

1. A *cycle* on $n \geq 3$ is a graph isomorphic to the graph:
   $C_n = (\{1, ..., n\}, \{\{i, i+1\};\ 1 \leq i \leq n-1\} \cup \{\{1, n\}\})$.
2.
   - The *length* of a path or a cycle is the number of its edges.
   - *k-path* (resp. *k-cycle*) is a path (resp. cycle) of length k.
   - A *k-path* (resp. *k-cycle*) is *odd* or *even* according to the parity of length k.
   - A 3-cycle is often called a *triangle*.

Remark 4.1
The cycle C_n is obtained from the path P_n by adding the edge $\{1, n\}$.

Examples:

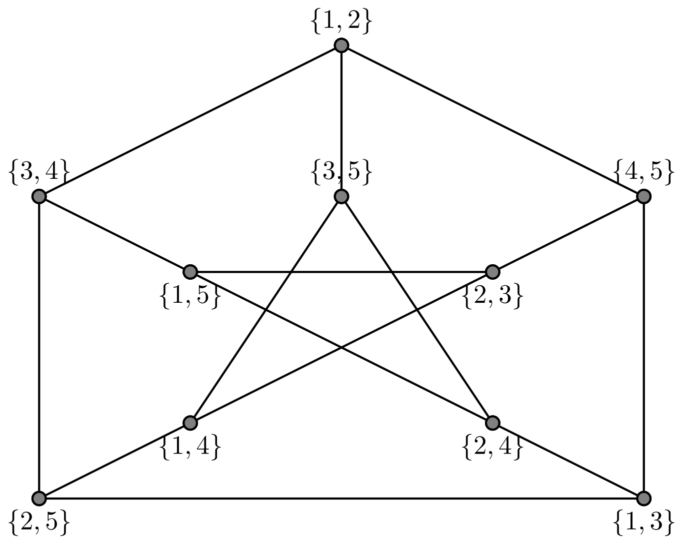


C_3 C_4 C_5

4.5 Petersen graph

A *Petersen graph* is a graph $G = (V, E)$, up to isomorphy, defined by:
$V = \mathcal{P}_2(\{1, 2, 3, 4, 5\}) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$
and for $i \neq j \in \{1, 2, 3, 4, 5\}$ and $\alpha \neq \beta \in \{1, 2, 3, 4, 5\}$ where:
$(\{\{i, j\}, \{\alpha, \beta\}\} \in E) \Leftrightarrow (\{i, j\} \cap \{\alpha, \beta\} = \emptyset)$

Diagram of Petersen graph

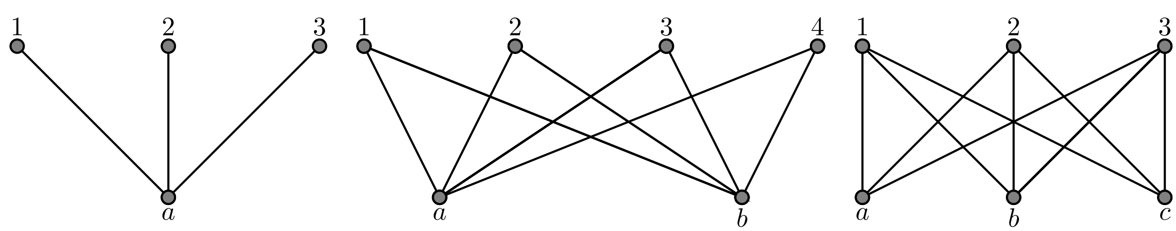


Petersen graph

4.6 Bipartite graphs

1.
   - A *Bipartite graph* is a graph $G = (V, E)$, such that V can be partitioned into two subsets X and Y such that every edge has one end in X and one in Y.
   - Such a partitioned $\{X, Y\}$ is called *a partition of the graph* G; X and Y are the parts of V, in this case G is denoted: $G[X, Y]$.
2.
   - If $G[X, Y]$ is a bipartite graph such that every $x \in X$ is joined to every $y \in Y$, then G is called a *Complete bipartite graph.*
   - Up to isomorphy, we denoted $K_{p,q}$ the complete bipartite graph $G[X, Y]$ with: $|X| = p$ and $|Y| = q$.

Examples:

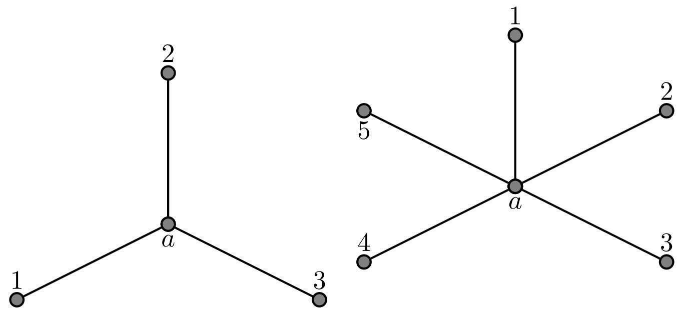


$K_{1,3}$ $K_{2,4}$ $K_{3,3}$

4.7 Star graphs

A *Star* is a complete bipartite graph $G[X, Y]$ with: $(|X| = 1$ or $|Y| = 1)$.

Examples:

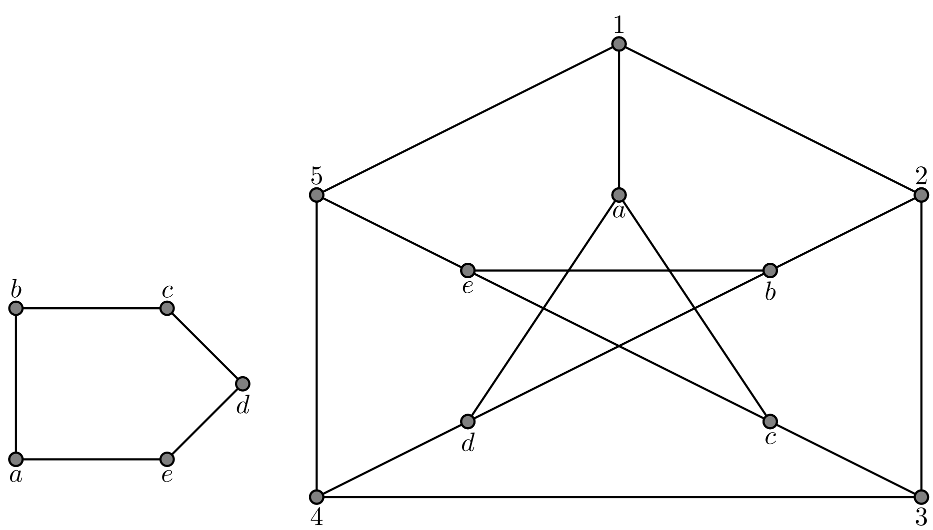


$K_{1,3}$ $K_{5,1}$

4.8 Regular graphs

- A *k-regular graph*, where $k \in \mathbb{N}$ is a graph $G = (V, E)$, such that: $\forall x \in V,\ d(x) = k$.
- A *regular graph* is a graph which is k-regular graph for some k.

Examples:



C_5 is a 2-regular graph A Patersen graph is 3-regular graph

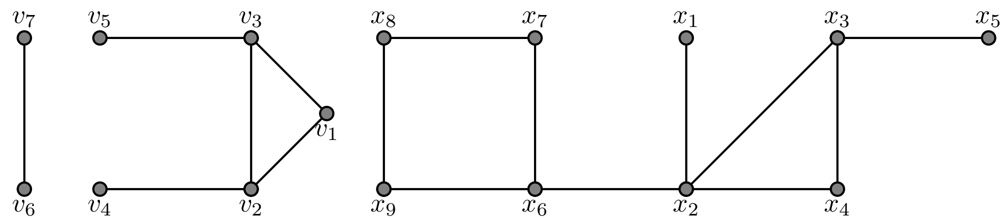
In general, C_n is a 2-regular graph.

4.9 Disconnected graphs

- A *disconnected graph* is a graph $G = (V, E)$, where V can be partitioned into $\{X, Y\}$ such that:$(X \neq \emptyset,\ Y \neq \emptyset,\ \forall (x, y) \in X \times Y : \{x, y\} \notin E)$.
- If a graph G is not disconnected, we say that G is *connected graph*.

Examples:

- Given a graph $G = (\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \{v_1v_2, v_2v_3, v_3v_1, v_2v_4, v_3v_5, v_6v_7\})$.
- Given a graph $H = (\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$,
  $\{x_1x_2, x_2x_3, x_3x_4, x_2x_4, x_2x_6, x_3x_5, x_6x_7, x_7x_8, x_8x_9, x_6x_9\})$.



G is disconnected graph $\qquad$ H is connected graph

5 Degree Sequence

5.1 Definition

Definition 5.1

We say that an increasing sequence $D = (d_1, ..., d_n)$ is graphic if there is a graph G having D as the degree sequence (i.e: $D = DEG(G)$).

Remarks 5.2

If an increasing sequence $D = (d_1, ..., d_n)$ is graphic, then

1. $d_n \leq n - 1$.
2. *D has an even odd terms.*
3. *If $d_1 = 0$, then $d_n \leq n - 2$.*
   If $d_n = n - 1$, then $d_1 \geq 1$.
4. *We remark there are $i \neq j$ such that $d_i = d_j$.*

Notation 5.3

1. *$D = (d_1, ..., d_n)$ an increasing sequence of integers with:*
   $0 < d_1 \leq ... \leq d_n < n$, where $n \geq 2$.
2. *$D^{''} = (d_1^{''}, ..., d_{n-1}^{''})$ the sequence obtaining, from D, as follows:*
   - *delete d_n from D and*
   - *Subtract 1 from each of the d_n last remaining terms.*
3. *$D^{'} = (d_1^{'}, ..., d_{n-1}^{'})$ the increasing sequence consists of integers $\{d_1^{''}, ..., d_{n-1}^{''}\}$ arranged in ascending order.*

5.2 Havel-Hakimi Theorem

Problematic:

A degree sequence can be obtained from graph. But how to get graph from degree sequence? There can be many graph from a degree sequence or there can not be any graph.
So, how to know if a degree sequence is a **graphic** sequence?
The solution is the Havel-Hakimi Theorem.

Theorem 5.4 Havel-Hakimi Theorem
The sequence D is graphic if and only if the sequence $D^{'}$ is graphic.

Consequence

1. This theorem reduces the study of D to the study $D^{'}$.
2. Thus, we have an algorithmic test to check whether D is graphic and to generate a graph whenever one exists.

We remark, this theorem is easily deduced from the following lemma.

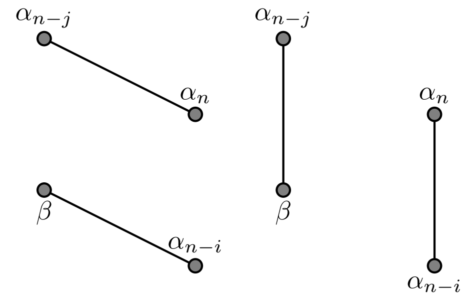
Lemma 5.5
Let $D = (d_1, ..., d_n)$ be a graphic sequence with: $d_n > 0$ (and then $n \geq 2$). Then there is a graph $G = (V, E)$ where, $V = \{x_1, ..., x_n\}$, such that:

- $\forall i \in \{1, ..., n\},\ d_G(x_i) = d_i,$ *and*
- $N_G(x_n) = \{x_{n-i};\ 1 \leq i \leq d_n\}$

Proof.

By contradiction.

1. Consider a graph $G = (\{\alpha_1, ..., \alpha_n\}, E)$ with:
   (a) $\forall i, d_G(\alpha_i) = d_i$
   (b) The cardinality $|N_G(\alpha_n) \cap \{\alpha_{n-i}; 1 \leq i \leq d_n\}|$ is **maximum**
   (for all graphs G with: $DEG(G) = D$).
2. So,
   (a) $\exists i, 1 \leq i \leq d_n, \{\alpha_{n-i}, \alpha_n\} \notin E$
   (b) $\exists j, d_n + 1 \leq j \leq n-1, \{\alpha_{n-j}, \alpha_n\} \in E$
   (c) We may assume that: $d_{n-j} < d_{n-i}$
   (d) As $\alpha_n \in N_G(\alpha_{n-j}) \backslash N_G(\alpha_{n-i})$, there are $\beta \neq \lambda \in N_G(\alpha_{n-i}) \backslash N_G(\alpha_{n-j})$, (with $\beta \neq \alpha_{n-j}$).
3. Thus, $\beta,\ \alpha_{n-j},\ \alpha_{n-i}$ and α_n are 4 distinct vertices of G, with: $\{\beta, \alpha_{n-i}\} \in E(G)$ and $\{\alpha_{n-j}, \alpha_n\} \in E(G)$; and $\{\beta, \alpha_n\}$ is an edge or not.
4. We consider the graph G' such that, $\beta,\ \alpha_{n-j},\ \alpha_{n-i}$ and α_n are 4 distinct vertices verifies of G', with $\{\beta, \alpha_{n-j}\} \in E(G')$ and $\{\alpha_{n-i}, \alpha_n\} \in E(G')$, the other edges are the same on G, hence, $DEG(G) = DEG(G') = D$, is a **contradiction** by the cardinality
   $|N_G(\alpha_n) \cap \{\alpha_{n-i}; 1 \leq i \leq d_n\}|$ is maximum.



$$G[\{\beta, \alpha_{n-j}, \alpha_{n-i}, \alpha_n\}] \quad G'[\{\beta, \alpha_{n-j}, \alpha_{n-i}, \alpha_n\}]$$

Algorithm of Havel-Hakimi

1. **Step** 1
   Sort the sequence in **increasing sequence** $D = (d_1, ..., d_n)$

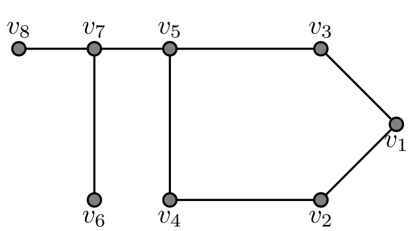
2. **Step** 2
   - Remove the term d_n in a sequence D.
   - Subtract 1 from each the d_n last terms in the sequence $(d_1, ..., d_{n-1})$.

3. **Step** 3
   - If a negative number in this new sequence, we stopped and the sequence $D = (d_1, ..., d_n)$ is not graphic.
   - If all number zeros in this new sequence, we stopped and the sequence $D = (d_1, ..., d_n)$ is graphic.
   - Otherwise, we arranged in ascending order this new sequence, consider $D' = (d_1', ..., d_{n-1}')$ the new increasing sequence obtained, and repeat from step 1.

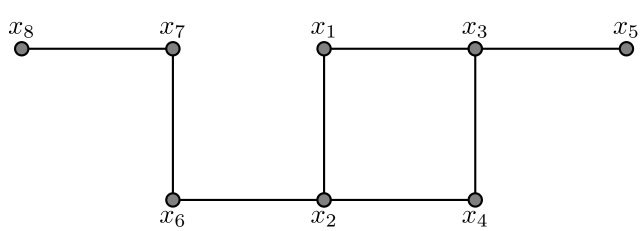
Example 5.6
Prove that the sequence $(1, 2, 3, 5, 3, 1, 2, 3)$ *is a graphic sequence and give an example of a graph* G *satisfying* $DEG(G) = D$.

Remark 5.7 *For the same degree sequence that is graphic, it is possible to find more than one graph which are not isomorphic.*

Example 5.8
Given a graph $G = (\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, \{v_1v_2, v_3v_1, v_2v_4, v_3v_5, , v_4v_5, v_5v_7, v_6v_7, v_7v_8\})$, *and a graph* $H = (\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \{x_1x_2, x_1x_3, x_3x_4, x_2x_4, x_2x_6, x_3x_5, x_6x_7, x_7x_8\})$.



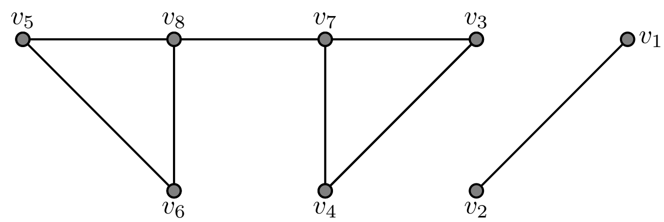
Graph G



Graph H

The two graphs G and H are not isomorphic, and they have the same degree sequence $(1, 1, 2, 2, 2, 2, 3, 3)$.

Using the Havel-Hakimi algorithm for the same degree sequence $(1, 1, 2, 2, 2, 2, 3, 3)$, *we find the graph* K *as follows:*



Graph K

The graphs K *and* G *are not isomorphic, and the graphs* K *and* H *are not isomorphic.*

Exercise 5.9

1. *Is the increasing sequence* $D = (d_1, ..., d_n)$ *a* **graphic** *sequence?*
   (a) $D = (1, 2, 3, 4, 4, 5, 6, 7)$
   (b) $D = (2, 2, 3, 3, 6, 6, 6, 6)$
   (c) $D = (2, 2, 3, 4, 4, 6, 6, 7)$
   (d) $D = (1, 1, 2, 4, 6, 7, 7, 8)$
2. *In the case, where* D *is graphic, give an example of graph* G *satisfying* $DEG(G) = D$.

6 Exercises of Graphs

Exercise 6.1

1. *If you pick five cards from a standard deck of* 52 *cards, then at least two will be of the same suit.*

2. *If you pick five numbers from the integers* 1 *to* 8*, then two of them must add up to nine.*

3. *In any group with two or more people, there must be at least two people who have the same number of friends.*
(Assume that "friend" is symmetric-if x is a friend of y, then y is a friend of x.)

4. *In a group of six people, there will always be three people that are mutual friends or mutual strangers.*
(Assume that "friend" is symmetric-if x is a friend of y, then y is a friend of x.)

Exercise 6.2

1. *Let G be a graph of order* 9 *such that each vertex has degree* 5 *or* 6*. Prove that at least five vertices have degree* 6 *or at least six vertices have degree* 5*.*

2. *Find the values of n such that path P_n (resp. cycle C_n) is self complementary.*

3. *Find a self complementary graph of order $4n$ (resp. $4n+1$) for each ($n \geq 1$).*

4. (a) *Show that every graph G has a path of length $\delta(G)$.*
   (b) *Show that if a graph G is connected and nontrivial graph, then the graph G is a complete graph or G has a path of length $\delta(G)+1$.*

5. *Prove that the* 4*-cube Q_4 is* 4*-regular, connected, and bipartite.*

Exercise 6.3

1. *Let $G[X,Y]$ be a bipartite graph, where $|X| = r \geq 1$ and $|Y| = s \geq 1$, of order n and size m.*
   (a) *Show that: $m \leq rs$.*
   (b) *Deduce that: $m \leq \frac{n^2}{4}$.*
   (c) *Describe the bipartite graphs G for which equality holds in question (b).*

2. *Let $G[X,Y]$ be a bipartite graph.*
   (a) *Show that: $\displaystyle\sum_{v\in X} d(v) = \sum_{v\in Y} d(v)$.*
   (b) *Deduce that if G is k-regular with $k \geq 1$, then $|X| = |Y|$.*

3. *Find a k-regular graph of order* 8 *for $k \in \{4,5\}$.*

4. *Consider two integers k and n with $k \geq 0$, $n \geq 1$. Prove the equivalence between the following two assertions.*

   (a) *There exists a k-regular graphs of order n.*
   (b) *The integer kn is even and $k \leq n-1$.*

5. *We define the graph is even if all its vertices are even.*
   Prove that the number of even graph on $\{1, 2, ..., n+1\}$ equals the number of graphs on $\{1, ..., n\}$.

Exercise 6.4

1. *Show that: a graph is bipartite if and only if it contains no odd cycle.*
2. *Let $G = (V, E)$ be a graph.*
   (a) *Show that if $\delta(G) \geq 2$, then G contains a cycle.*
   (b) *Show that if G is a simple graph and $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G) + 1$.*

Exercise 6.5

1. (a) *Is the increasing sequence $D = (d_1, ..., d_n)$ a* **graphic** *sequence?*
      i. $D = (1, 2, 3, 4, 4, 5, 6, 7)$
      ii. $D = (2, 2, 3, 3, 6, 6, 6, 6)$
      iii. $D = (2, 2, 3, 4, 4, 6, 6, 7)$
      iv. $D = (1, 1, 2, 4, 6, 7, 7, 8)$
   (b) *In the case, where D is graphic, give an example of graph G satisfying $DEG(G) = D$.*
2. *Consider the two sequences:* $D_1 = (1, 1, 3, 3, 3, 3, 5, 6, 8, 9)$
   and $D_2 = (3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6, 6, 6)$
   (a) *Show that: D_1 is not graphic.*
   (b) *Show that: D_2 is graphic and give a graph G with $DEG(G) = D_2$.*
   (c) *Show that there is no bipartite graph G such that $DEG(G) = D_2$.*

Exercise 6.6

1. *If $n+1$ numbers are selected from the set $\{1, 2, ..., 2n\}$, then one will divide another evenly.*
2. *Given a sequence $a_1, a_2, ..., a_{n^2+1}$ of any n^2+1 different numbers, there is either an increasing subsequence of $(n+1)$ terms or else a decreasing subsequence of $(n+1)$ terms.*

Exercise 6.7

1. *In any group of 6 people there are either three mutual friends or else three mutual strangers.*
2. *In any set of 10 people there is either a set of three mutual strangers or four mutual friends.*
3. *In any set of 10 people there is either a set of three mutual friends or a four mutual strangers.*
4. *In any set of 20 people there is either a set of four mutual friends or a four mutual strangers.*

Exercise 6.8

Let $G = (V, E)$ be a graph, with $|V| = 2n$, without triangles (i.e. there is no subset $X = \{a, b, c\}$ of V such that: $\{a, b\}, \{b, c\}, \{c, a\} \in E$).

1. (a) *Find the value of $|E|$ when G is n-regular.*
   (b) *Find an example where G is n-regular.*
2. *Assume that G is not n-regular.*
   (a) *Show that: $(\forall x \in V,\ d(x) \leq n) \Rightarrow (|E| < n^2)$.*
   (b) *Show that: $|E| < n^2$.*

Exercise 6.9

Find all graph self complementary, up to isomorphy, of order 5 or less.

Exercise 6.10

We know that a graph with $n \geq 2$ vertices has at least one pair of vertices of equal degrees. Find all graphs with exactly one pair of vertices with equal degrees. What are their degree sequences?

(Hint: Begin with $n \in \{2, 3, 4\}$. Use a recursive construction. Can degree 0 or $n - 1$ occur twice?)