With Complete Solution

318

Final Exam, November 2022

3 hours

Simple calculators are allowed

I) i) Find a general solution of the differential equation $(x+1)\dfrac{dy}{dx}+(x+2)y=2xe^{-x}$, $x>-1$. (4 points)

ii) Show that $\mu(y)=y^3$ is an integrating factor of the differential equation $xydx+(2x^2+3y^2-20)dy=0$ and solve the equation. (4 points)

II) i) Find a general solution of the differential equation $y''-16y=2e^{4x}$. (4 points)

ii) Find the form of a particular solution of the differential equation $y^{(4)}-y''=x+2xe^{x}-3xe^{-x}$ (Do not evaluate the coefficients). (4 points)

iii) Bacteria in culture grow at a rate proportional to the number of bacteria present at any time. Initial number of bacteria is 2000 and it is increased by 50% in 10 hours. What will be the number of bacteria in 20 hours? (4 points)

III) i) Find a general solution of the differential equation $xy''+(1-2x)y'+(x-1)y=0$, $x>0$, knowing that $y_1=e^x$ is a particular solution of the given equation. (4 points)

ii) Find the solution of the initial value problem $x^2y''+3xy'+2y=0$, $y(1)=0$, $y'(1)=1$. (4 points)

IV) Find a general solution to the differential equation $y''-2y'+y=\dfrac{e^x}{x}$, $x>0$. (6 points)

V) Let $f(x)=\begin{cases} 0, & -\pi<x<-\frac{\pi}{2} \\ -1, & -\frac{\pi}{2}\le x<0 \\ 0, & x=0 \\ 1, & 0<x\le\frac{\pi}{2} \\ 0, & \frac{\pi}{2}<x<\pi \end{cases}$

i) Sketch the function f on $(-\pi,\pi)$. (2 points)

ii) Compute the Fourier series of f on $(-\pi,\pi)$. (4 points)

Complete solution of the final Exam 318 M.
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Quest ① (I)

④ ⓐ ⓒ $(x+1)y' + (x+2)y = 2xe^{-x}$ is Linear D.E.

① $y' + \frac{x+2}{x+1} = \frac{2}{x+1} x e^{-x}$, $P(x) = \frac{x+1+1}{x+1} = 1 + \frac{1}{x+1}$

② $\mu(x) = e^{\int P(x)dx} = e^{\int(1+\frac{1}{x+1})dx} = e^{x+\ln(x+1)} = (x+1)e^{x}$

$y(x+1)e^{x} = y\mu(x) = \int \frac{2}{x+1} x e^{-x} e^{x}(1+x) = \int 2x\,dx = x^2 + c$

① $\boxed{y = \frac{c}{x+1}e^{-x} + \frac{x^2 e^{-x}}{x+1}}$

④ ⓑ ⓘⓘ $y^3\ (xy\,dx + (x^2 + 3y^2 - 20)dy = 0)$

$\underbrace{xy^4}_{M} dx + \underbrace{(2x^2y^3 + 3y^5 - 20y^3)}_{N} dy = 0$

① $\frac{\partial M}{\partial y} = 4xy^3 = \frac{\partial N}{\partial x} \Rightarrow \exists F$ of x and y s.t

$\frac{\partial F}{\partial x} = M = xy^4$, $\frac{\partial F}{\partial y} = 2x^2y^3 + 3y^5 - 20y^3$

① $\int \frac{\partial F}{\partial x}dx = F(x,y) = \int xy^4 dx = \frac{1}{2}x^2y^4 + \phi(y)$

$\frac{\partial F}{\partial y} = \cancel{2x^2y^3} + \phi'(y) = \cancel{2x^2y^3} + 3y^5 - 20y^3$

① $\phi'(y) = 3y^5 - 20y^3 \Rightarrow \phi(y) = \frac{1}{2}y^6 - 5y^4 + c$

Then, the solution of the D.E is

① $\boxed{F(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + c = 0}$

Question II (2)

④ ⓘ ⓐ $y'' - 16y = 2e^{4x}$, $y = e^{mx}$

② 1) $y'' - 16y = 0 \Rightarrow m^2 - 16 = 0 \Rightarrow m = \mp 4$

$y_c = c_1 e^{4x} + c_2 e^{-4x}$

① 2) $y_p = Axe^{4x}$, $y_p' = Ae^{4x} + 4Axe^{4x}$

$y_p'' = 8Ae^{4x} + 16Axe^{4x}$

$$y_p'' - 16y_p = 8Ae^{4x} + 16Axe^{4x} - \cancel{16Axe^{4x}} = 2e^{4x}$$

$$\Rightarrow 8A = 2 \Rightarrow A = \frac{1}{4}$$

① $y_p = \frac{1}{4}xe^{4x}$

Then the G. solution of the D.E is

$$\boxed{y = y_c + y_p = c_1e^{4x} + c_2e^{-4x} + \frac{1}{4}xe^{4x}}$$

④ (ii)(b) $y^{4} - y'' = x + 2xe^{x} - 3xe^{-x}$

i) $m^4 - m^2 = m^2(m^2-1) = m^2(m-1)(m+1) = 0$

② $m = 0, 0, \ m = 1, \ m = -1$

$$y_c = c_1 + c_2x + c_3e^{x} + c_4e^{-x}$$

② $\boxed{y_p = (A_1 + A_2x)x^2 + (B_1 + B_2x)xe^{x} + (C_1 + C_2x)xe^{-x}}$

④ (ii)(c) $P(t) = ce^{kt}, \ P(0) = 2000$

$P(10) = 2000 + 1000 = 3000$

② $2000 = P(0) = c \Rightarrow$

$$P(t) = 2000e^{kt}$$

$$P(10) = 2000e^{10k} = 3000$$

$$\frac{3}{2} = e^{10k} \Rightarrow \ln\left(\frac{3}{2}\right) = 10k$$

① $k = \frac{1}{10}\ln\left(\frac{3}{2}\right)$

$$P(t) = 2000e^{\frac{1}{10}\ln(3/2)t}$$

$$P(20) = 2000e^{\frac{1}{10}\ln\left(\frac{3}{2}\right)20} = 2000e^{2\ln\left(\frac{3}{2}\right)} = e^{\ln\left(\frac{3}{2}\right)^2} = \frac{9}{4}(\cancel{2000}^{500})$$

$\boxed{P(20) = 4500}$ ①

Question III (3) $xy'' - (x+1)y' + y = 0, \ x > 0 \quad y_1 = e^{x}$

④ (i)(a) ① $y'' - \frac{x+1}{x}y' + \frac{1}{x}y = 0, \ P(x) = -\left(1 + \frac{1}{x}\right)$

② $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2}dx, \quad e^{-\int -\left(1+\frac{1}{x}\right)dx} = e^{x + \ln x} = e^{x}x$

$$y_2 = e^{x}\int \frac{e^{x}x}{e^{2x}}dx = e^{x}\int xe^{-x}dx = e^{x}\left[-xe^{-x} - e^{-x}\right] = -x - 1 = -(x+1)$$

① $\boxed{y_2 = -(x+1)}$ or $\boxed{y_2 = x+1}$

So the G. solution of the D.E. is

$$\boxed{y = c_1 e^x + c_2 (x+1)}$$

(ii) (b) $\begin{cases} x^2 y'' + 3x y' + 2y = 0 \\ y(1) = 0, \quad y'(1) = 1 \end{cases} \qquad y = x^m, \quad x > 0$

(4)

$$m(m-1) + 3m + 2 = m^2 + 2m + 2 = 0$$

$$(m+1)^2 + 1 = 0, \quad (m+1)^2 = +i^2$$

$$|m+1| = i, \quad m+1 = \mp i, \text{ hence } \boxed{m = -1 \mp i} \quad 1$$

(1) $y = c_1 x^{-1} \cos(\ln x) + c_2 x^{-1} \sin(\ln x)$

$$y' = -c_1 x^{-2} \cos(\ln x) + c_1 x^{-1} (-\sin(\ln x)) \frac{1}{x}$$
$$- c_2 x^{-2} \sin(\ln x) + c_2 x^{-1} \cos(\ln x) \left(\frac{1}{x}\right)$$

$$y(1) = \underline{c_1 = 0}, \quad y'(1) = -c_1 + c_2 = 1 \Rightarrow \underline{c_2 = 1}$$

So the unique solution of the IVP is given by

(2) $$\boxed{y(x) = x^{-1} \sin(\ln x)}$$

Question IV(4) $\quad y'' - 2y' + y = \frac{e^x}{x}; \quad x > 0$

(6)

1) $y'' - 2y' + y = 0, \quad y = e^{mx}, \quad m^2 - 2m + 1 = (m-1)^2 = 0, \quad m = 1, 1$

(2) $y_c = c_1 e^x + c_2 x e^x, \quad y_1 = e^x, \quad y_2 = x e^x$

2) $y_p = y_1 u_1 + y_2 u_2$, where u_1, u_2 satisfying two equations

$$\begin{cases} u_1' e^x + u_2' (x e^x) = 0 \\ u_1' e^x + u_2' (e^x + x e^x) = \frac{e^x}{x} \Big/ 1 \end{cases}$$

We have
$$u_1' + x u_2' = 0$$
$$u_1' + u_2'(1+x) = \frac{1}{x}$$
$$W = \begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = 1$$

$$u_1' = \frac{\begin{vmatrix} 0 & x \\ 1/x & 1+x \end{vmatrix}}{W} = \frac{-1}{1} = -1 \Rightarrow \boxed{u_1 = -x} \quad 1\tfrac{1}{2}$$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 1 & 1/x \end{vmatrix}}{W} = \frac{1/x}{1} = \frac{1}{x} \Rightarrow \boxed{u_2 = \ln x} \quad 1\tfrac{1}{2}$$

$$y_p = -x e^x + x e^x \ln x = \boxed{x e^x (\ln x - 1)}$$

Then the G. solution of the D.E is

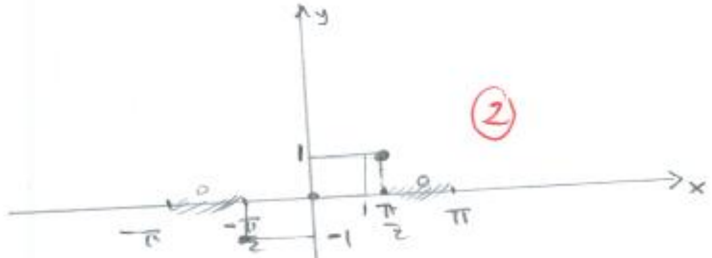
$$\boxed{y = y_c + y_p = c_1 e^{x} + c_2 x e^{x} + x e^{x}(\ln x - 1)} \quad 1$$

Question V (5)

$$f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ -1 & -\frac{\pi}{2} < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

1) We sketch the function on $(-\pi, \pi)$

2



4 2) This function is odd on $(-\pi, \pi) \Rightarrow a_0 = a_n = 0,\ n = 1, 2, \dots$ 1

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{\pi}\right) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

2

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (1)\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0\,dx$$

$$= \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi/2}$$

$$\boxed{b_n = \frac{2}{\pi n}\left(1 - \cos(n\pi/2)\right)}$$

$$\frac{f(x^+) + f(x^-)}{2} = \sum_1^{\infty} b_n \sin(nx) = \boxed{\sum_1 \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)\sin(nx) \quad -\pi < x < \pi}$$

1

Results out of the Question V At $x = \frac{\pi}{2}$, we deduce نتيجة خارج عن الامتحان

$$\frac{f\left(\frac{\pi}{2}^+\right) + f\left(\frac{\pi}{2}^-\right)}{2} = \frac{0+1}{2} = \frac{1}{2} = \sum_1^{\infty} \frac{2}{n\pi}\left(1 - \cos(n\pi/2)\right)\sin(n\pi/2)$$

$$\frac{1}{2} = \sum_1^{\infty} \frac{2}{(2n-1)\pi}\left(1 - \cancel{\cos\left(\frac{2n-1}{2}\pi\right)}^{\,0}\right)\underbrace{\sin\left(\frac{2n-1}{2}\pi\right)}_{(-1)^{n-1}}$$

$$\boxed{\frac{\pi}{4} = \sum_1^{\infty} \frac{(-1)^{n-1}}{2n-1}}$$

$$\boxed{\frac{f(x^+) + f(x^-)}{2} = \frac{2}{\pi}\left[\sin x + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5}{5} + \frac{2}{6}\sin 6x + \cdots\right.}$$

4

Question ③

(ii) (a) $x y'' + (1-2x) y' + (x-1) y = 0, \quad x > 0$, where $y_1 = e^x$ is a Particular Solution of the D.E

① $y'' + \left(\frac{1}{x} - 2\right) y' + \frac{x-1}{x} y = 0, \quad P(x) = \left(\frac{1}{x} - 2\right)$

② $e^{-\int P(x)dx} = e^{\int (2 - \frac{1}{x}) dx} = e^{2x - \ln x} = e^{2x} \cdot e^{\ln x^{-1}}$
$= \frac{1}{x} e^{2x}$

① $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx = e^x \int \frac{\frac{1}{x} e^{2x}}{e^{2x}} dx$
$= e^x \int \frac{dx}{x} = e^x \ln x$

Then $y = c_1 e^x + c_2 e^x \ln x$ is the G. Solution of the D.E