Introduction to Real Analysis

Continuity

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Continuous Functions

Definition

If $f : D \longrightarrow \mathbb{R}$ and $c \in D$, then f f is continuous at c if

$$\forall \varepsilon > 0 \;\; \exists\, \delta > 0 : \quad x \in D, \quad |x - c| < \delta \quad \Rightarrow |f(x) - f(c)| < \varepsilon$$

Continuous Functions

Theorem

A function $f : D \longrightarrow \mathbb{R}$ is continuous at $c \in D$ iff for every sequence (x_n) in D such that $x_n \longrightarrow c$ then $(f(x_n))$ converges to $f(c)$.

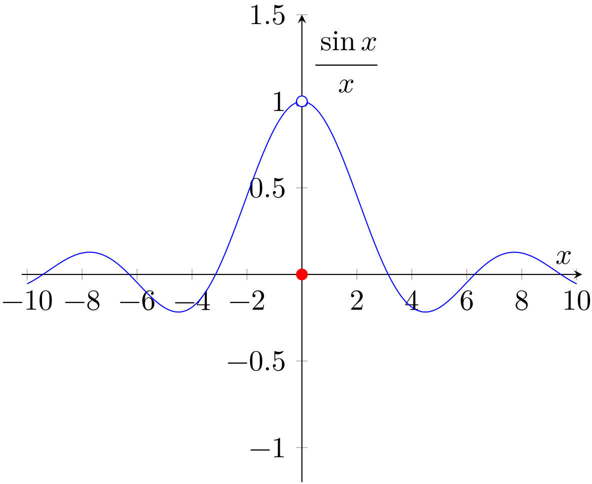
Continuous Functions

Examples

1. Polynomials are continuous.
2. The fucntion $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Continuous Functions

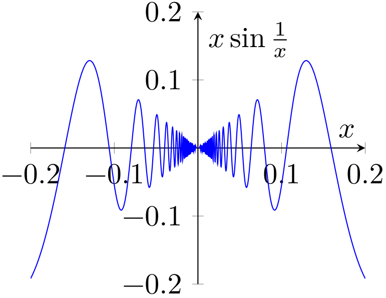


Continuous Functions

Examples

If $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



Combination of Continuous Functions

Theorem

If $f, g : D \longrightarrow \mathbb{R}$ are continuous at $c \in D$ then $f+g$ and fg are continuous at c
and f/g is continuous at c if $g(c) \neq 0$.

Continuous Functions

Examples

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$
2. $f(x) = \dfrac{x + 4}{x^4 + 2}$
3. $\sin x, \cos x$
4. $\tan x = \dfrac{\sin x}{\cos x}$
5. $\cot x = \dfrac{\cos x}{\sin x}$

Composite Function

Theorem

Let $f : D \longrightarrow \mathbb{R}$, $g : E \longrightarrow \mathbb{R}$ and $f(D) \subset E$. If f is continuous at $c \in D$ and g is continuous at $f(c)$ then the composite function $g \circ f : D \longrightarrow \mathbb{R}$ is continuous at c.

Continuous Functions

Examples

1. If $f : D \longrightarrow \mathbb{R}$ is continuous, then $|f|$ is continuous on D.
2. If $f : D \longrightarrow [0, \infty)$ is continuous, then $\sqrt{f}$ is continuous on D.

Continuity on an Interval

Definition

A function $f : D \longrightarrow \mathbb{R}$ is bounded if there is $M > 0$ such that

$$|f(x)| \leq M \qquad \forall x \in D$$

Bounded Functions

Theorem

If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous then it is bounded.

Maximum and Minimum

Definition

The function $f : D \longrightarrow \mathbb{R}$ is said to have a minimum on D if there is $x_1 \in D$ such that

$$f(x_1) \leq f(x) \quad \forall x \in D$$

and a maximum on D if there is $x_2 \in D$ such that

$$f(x) \leq f(x_2) \quad \forall x \in D$$

Continuity on an Interval

Examples

1. $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = x^2$$

2. $f : (0, 1) \longrightarrow \mathbb{R}$

$$f(x) = x^2$$

3. $f : [0, 1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1/x & x \neq 0 \\ 2 & x = 0 \end{cases}$$

Continuity on an Interval

Theorem

If I is a closed and bounded set and $f : I \longrightarrow \mathbb{R}$ is continuous then f has a maximum and a minimum on I.

Intermediate Value Theorem

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. If λ is a real number between $f(a)$ and $f(b)$ then there is $c \in (a, b)$ such that $f(c) = \lambda$.

Continuity on an Interval

Corollary

If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $f(a)$, $f(b)$ have opposite signs then there is a point $c \in (a, b)$ such that $f(c) = 0$.

Examples

Show that $x^3 - 2 = 0$ has a solution in $[1, 2]$.

Continuity on an Interval

Corollary

If I is an interval, and $f : I \longrightarrow \mathbb{R}$ is continuous then $f(I)$ is an interval.

Continuity on an Interval

Corollary

If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous then $f([a, b])$ is a closed and bounded interval.

Fixed Point

Examples

If $f : [0, 1] \longrightarrow [0, 1]$ is continuous then show that f has a fixed point $c \in [0, 1]$, i.e., $f(c) = c$.

Continuity on an Interval

Theorem

If I is an interval and $f : I \longrightarrow \mathbb{R}$ is continuous and injective then it is strictly monotonic.

Theorem

If f is a monotonic function on the interval I and $f(I)$ is an interval then f is continuous.

Continuity on an Interval

Theorem

If I is an interval and $f : I \longrightarrow \mathbb{R}$ is continuous and injective then f^{-1} is continuous and strictly monotonic.

Continuity on an Interval

Examples

Prove that $f(x) = \sqrt[n]{x}$ is continuous on $[0, \infty)$.