Chapter 8
Analysis of Variance (ANOVA)

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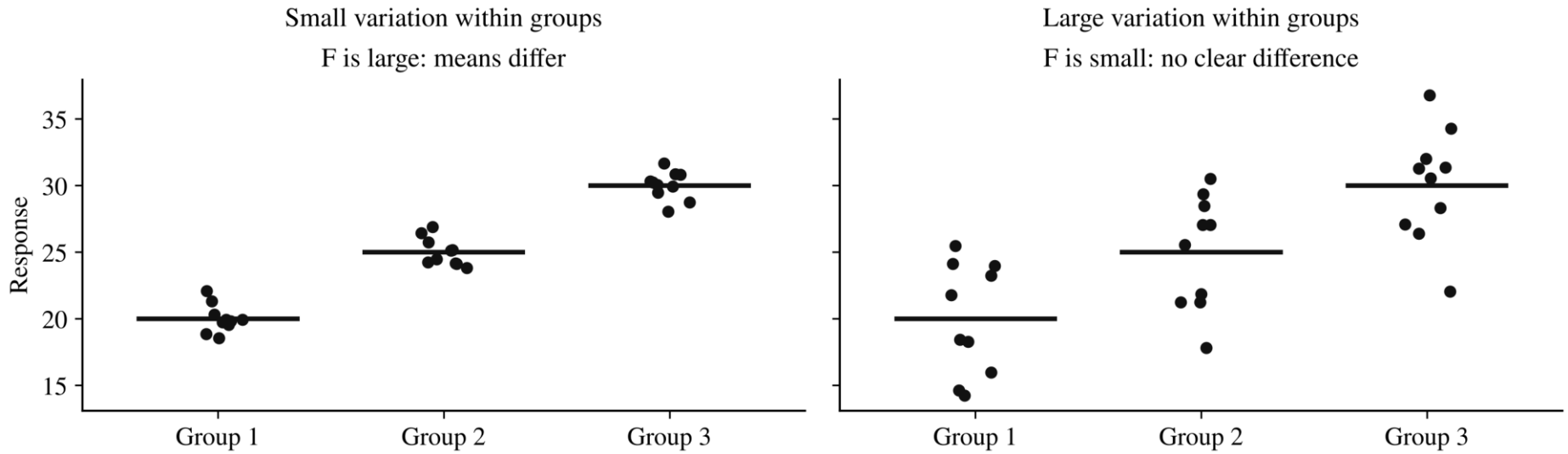
Basic Ideas of ANOVA

What ANOVA does · terms · assumptions · hypotheses

What Is ANOVA?

The **analysis of variance (ANOVA)** tests the claim that **three or more** populations means are equal. It does this by comparing the variation **between** the samples with the variation **within** each sample

ANOVA is an extension of the two independent samples t test



Idea

If the variation between groups is much larger than the variation within groups, the means are different

If the variation between groups and the variation within groups are about the same size, there is no clear difference between the means.

Terms Used in ANOVA

1 Response (what we measure)

The dependent variable. It is measured on an interval or ratio scale

2 Factor

The independent variable. It is nominal or ordinal with more than 2 categories. One-way ANOVA has one factor and two-way ANOVA has two

3 Levels

The possible values of a factor. In one-way ANOVA they are also called treatments

4 Unit (who or what we measure from)

The person, animal or piece of material that gets a treatment and gives a response

Remember

One-way ANOVA = one factor = one independent variable with two or more levels

Assumptions of One-Way ANOVA

1. All populations follow a **normal distribution**
2. All populations have the **same variance** σ^2
3. The samples are **random** and **independent** of each other

Variables

The response is **quantitative** and the factor is **qualitative**

Parametric test

ANOVA needs normal populations. If this is not true we use a non-parametric test

Hypotheses of One-Way ANOVA

1 Null hypothesis

$$H_0: \ \mu_1 = \mu_2 = \cdots = \mu_k$$

All k treatment means are equal. The treatments have no different effect on the response

2 Alternative hypothesis

$$H_1: \ \text{at least one mean is different}$$

Note

ANOVA does not say which mean is different or which one is larger. It only says that not all means are equal

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One-Way ANOVA

Model · sums of squares · F test · ANOVA table

The One-Way ANOVA Model

$$y_{ij} = \mu_i + e_{ij}$$

- $\boldsymbol{i} = 1, \ldots, k$ is the treatment number.
- $\boldsymbol{j} = 1, \ldots, n_i$ is the unit number Inside treatment i .
- n_i the number of Unit in treatment i .
- $n = n_1 + n_2 + \ldots + nk$ is the total number of observations.
- y_{ij} is the response of unit j in treatment i
- μ_i is the unknown population mean of treatment i
- e_{ij} are independent random errors with $N(0, \sigma^2)$

Data Layout

Treatment	1	2	…	k	
Observations	y_{11}	y_{21}	…	y_{k1}	
	y_{12}	y_{22}	…	y_{k2}	
	⋮	⋮	⋮	⋮	
	y_{1n_1}	y_{2n_2}	…	y_{kn_k}	
Total	$Y_{1\,.}$	$Y_{2\,.}$	…	$Y_{K\,.}$	$Y_{..}$
Sample mean	$\overline{y}_{1.}$	$\overline{y}_{2.}$	…	$\overline{y}_{k.}$	$\overline{Y}_{..}$
Sample variance	${S_1}^2$	${S_2}^2$	…	${S_k}^2$	

The F-Ratio

ANOVA compares two kinds of variation

Between groups
Differences between the group means. It is large when the means are far apart.

Within groups
Sampling error and differences between units in the same group. It is the error variation

$$F = \frac{\text{variance between groups}}{\text{variance within groups}}$$

Rule

Small F: the groups are probably not different (groups are not significantly different from each other).
Large F: two or more groups are probably different.
Use the F table to decide .

Steps of One-Way ANOVA

1 State the hypotheses

Write H_0 and H_1

2 Compute SS

Sums of squares SSA, SSE and SST

3 Compute df

Degrees of freedom for each source

4 Compute MS

Mean squares MSA and MSE

5 Compute F

F = MSA / MSE

6 Take a decision

Compare F with the critical value or use the p-value

The Grand Mean

The **grand mean** is the average of all **n** observations when the factor is ignored. It is a weighted average of the sample means

$$\bar{y}_{..} = \frac{\sum_{i=1}^{k} n_i \bar{y}_{i.}}{\sum_{i=1}^{k} n_i} = \frac{n_1\bar{y}_{1.} + n_2\bar{y}_{2.} + \cdots + n_k\bar{y}_{k.}}{n_1 + n_2 + \cdots + n_k}$$

$$\bar{y}_{i.} = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$$

Note

$\bar{y}_{i.}$ is the mean of treatment i

$\bar{y}_{..}$ is the mean of all observations

Sums of Squares

SST
Total variation
(total sum of squares)

$$\text{SST} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij} - \bar{y}_{..}\right)^2$$

SSA
Between groups Variation
(treatment sum of squares)

$$\text{SSA} = \sum_{i=1}^{k} n_i\left(\bar{y}_{i.} - \bar{y}_{..}\right)^2$$

SSE
Within groups
(error sum of squares)

$$\text{SSE} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij} - \bar{y}_{i.}\right)^2 = \sum_{i=1}^{k}(n_i - 1){S_i}^2$$

$$\text{SST} = \text{SSA} + \text{SSE}$$

SSA is also written SS(B) and SSE is also written SS(W) in some notes

Degrees of Freedom

A **degree of freedom** is a value that is free to vary. Example: six numbers with mean 40 must add to 240. Five of them can be anything and the last one is fixed, so df = 6 − 1 = 5

Between groups
One less than the number of groups

$$\mathrm{df}(A) = k - 1$$

Within groups

$$\mathrm{df}(\mathrm{E}) = n - k$$

Total
One less than the sample size

$$\mathrm{df}(Total) = n - 1$$

Check
(k − 1) + (n − k) = n − 1. If all samples have size n_i then $df\,(E) = k(n_i - 1)$

Mean Squares

A **mean square** (MS) is a sum of squares divided by its degrees of freedom. It is an average squared deviation

$$\mathrm{MS} = \frac{\mathrm{SS}}{\mathrm{df}}$$

1. **Between groups**
   (Mean square for treatments)

$$\mathrm{MSA} = \frac{\mathrm{SSA}}{k-1}$$

2. **Within groups**
   (Mean square for error)

$$\mathrm{MSE} = \frac{\mathrm{SSE}}{n-k}$$

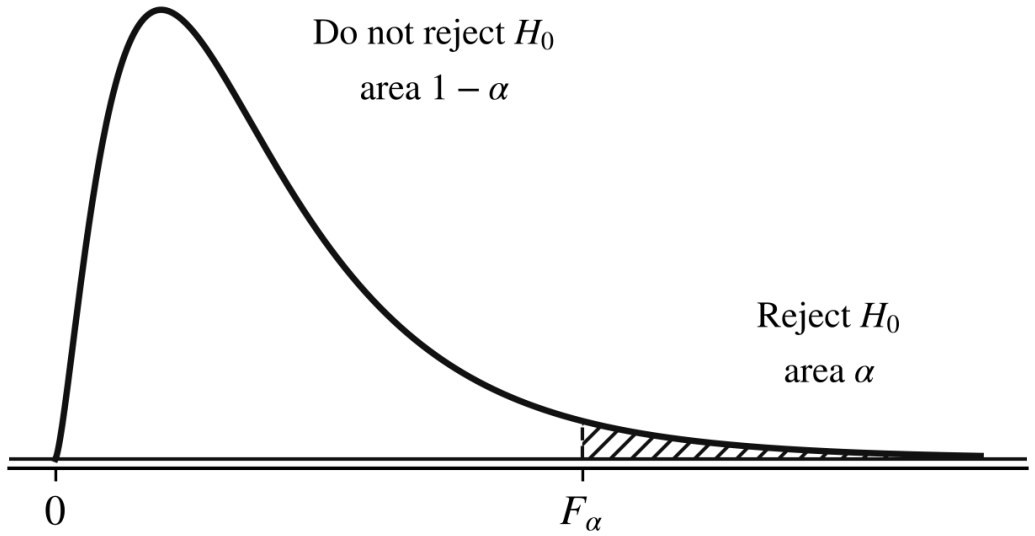
Notes

MSE is the **pooled estimate** of the common variance σ^2.

The F Test

$$F_{\text{TS}} = \frac{\text{MSA}}{\text{MSE}}$$

- F_{TS} has an **F** distribution with **k – 1** (numerator) and **n – k** (denominator) degrees of freedom.
- The **F** test is a **right tail** test.



Why right tail

A large F means the group means are far apart compared with the variation inside the groups

Decision

Reject H_0 if $F_{TS} > F_\alpha(k-1, n-k)$

The ANOVA Table

Source	SS	df	MS	F
Between (A)	SSA	$k-1$	$\text{MSA} = \frac{\text{SSA}}{k-1}$	$F_{\text{TS}} = \frac{\text{MSA}}{\text{MSE}}$
Within (E)	SSE	$n-k$	$\text{MSE} = \frac{\text{SSE}}{n-k}$	
Total	SST	$n-1$		

Example 1: One-Way ANOVA

Moisture absorption of five concrete aggregates

Example 1

An engineer wants to know if the mean absorption of moisture in concrete is different among 5 concrete aggregates. The samples are exposed to moisture for 48 hours. Six observations are tested for each aggregate, so 30 samples in total.

Aggregate	Aggregate 1	Aggregate 2	Aggregate 3	Aggregate 4	Aggregate 5	All
1	551	595	639	417	563	
2	457	580	615	449	631	
3	450	508	511	517	522	
4	731	583	573	438	613	
5	499	633	648	415	656	
6	632	517	677	555	679	
Total ($Y_{1.}$)	**3320**	**3416**	**3663**	**2791**	**3664**	$Y_{..} = \mathbf{16854}$
Mean ($\overline{y}_{i.}$)	**553.33**	**569.33**	**610.5**	**465.17**	**610.67**	$\overline{Y}_{..} = \mathbf{561.8}$
Variance (S_i^2)	**12133.87**	**2302.67**	**3593.50**	**3318.57**	**3455.47**	

? At the 0.05 level of significance, test if the mean absorption is the same for the 5 aggregates

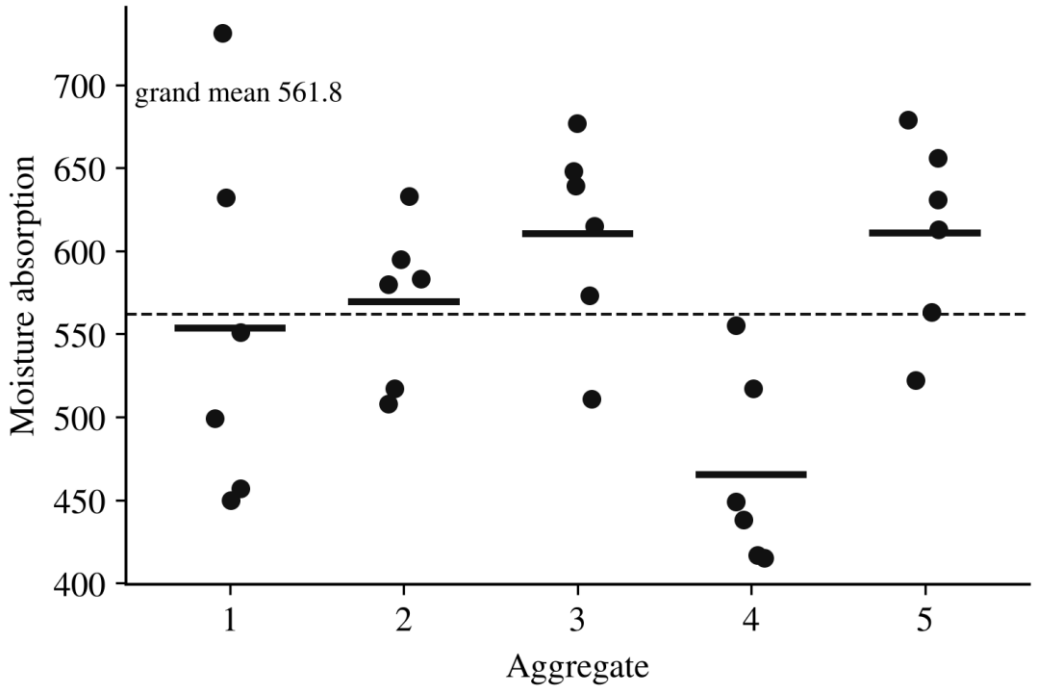
Example 1: Step 1 Hypotheses

1 Hypotheses

$$H_0: \ \mu_1 = \mu_2 = \cdots = \mu_5$$

H_1: at least one mean is different

	1	2	3	4	5
Size (n_i)	6	6	6	6	6
Mean ($\overline{y}_{i.}$)	553.33	569.33	610.50	465.17	610.67
St. Dev. (S_i)	110.15	47.99	59.95	57.61	58.78
Variance S_i^2	12133.87	2302.67	3593.50	3318.57	3455.47



Example 1: Step 2 Sums of Squares

Grand mean
Mean of all 30

$$\bar{y}_{..} = \frac{3320 + 3416 + 3663 + 2791 + 3664}{30} = \frac{16854}{30} = 561.8$$

SSA
Between concrete aggregates

$$\text{SSA} = \sum_{i=1}^{k} n_i \left(\bar{y}_{i.} - \bar{y}_{..}\right)^2 = 6\left((553.33 - 561.8)^2 + \cdots + (610.67 - 561.8)^2\right) = 85356.47$$

SSE
Within concrete aggregates

$$\text{SSE} = \sum_{i=1}^{k} (n_i - 1) {S_i}^2 = 5(12133.87 + 2302.67 + 3593.50 + 3318.57 + 3455.47) = 124020.33$$

SST
Total

$$\text{SST} = \text{SSA} + \text{SSE} = 85356.47 + 124020.33 = 209376.80$$

The formulas show the means and variances rounded to 2 decimals.

Example 1: Steps 3 and 4

3 Degrees of freedom

$$\mathrm{df}(A) = 5 - 1 = 4$$

$$\mathrm{df}(\mathrm{E}) = 30 - 5 = 25$$

$$\mathrm{df}(Total) = 30 - 1 = 29$$

4 Mean squares

$$\mathrm{MSA} = \frac{85356.47}{4} = 21339.12$$

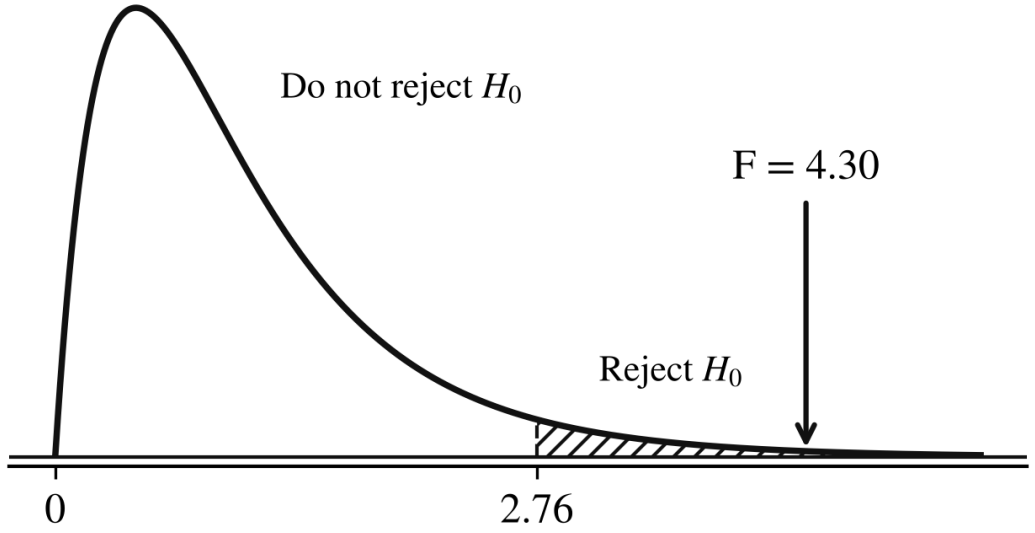
$$\mathrm{MSE} = \frac{124020.33}{25} = 4960.81$$

Example 1: Step 5 F Test

5 F statistic

$$F = \frac{\text{MSA}}{\text{MSE}} = \frac{21339.12}{4960.81} = 4.30$$

- $\alpha = 0.05$, df = (4, 25)
- Critical value $F_{0.05}(4, 25) = 2.76$ from the F table
- The p-value is $P(F > 4.30) \approx 0.009$ from software



Decision

F = 4.30 > 2.76, so we **reject** H_0.

The p-value 0.009 is less than 0.05

Example 1: Step 6 Decision

Source	SS	df	MS	F	Critical F	p-value
Between aggregates	85356.47	4	21339.12	4.30	2.76	0.009
Within aggregates	124020.33	25	4960.81			
Total	209376.80	29				

Decision

F = 4.30 is greater than 2.76 and the p-value 0.009 is less than 0.05. We reject H_0

Conclusion

The aggregates do not have the same mean absorption of moisture. To find which aggregates differ we need confidence intervals or post hoc tests (not in this course)

Example 2: One-Way ANOVA

Example 2

An instructor divided the class into three rows: first (F), second (S) and third (T). A random sample of students in each row was taken. The table shows their scores on the second exam

Row	1	2	3	4	5	6	7	8	9
First (F)	82	83	97	93	55	67	53		
Second (S)	83	78	68	61	77	54	69	51	63
Third (T)	38	59	55	66	45	52	52	61	

? At the 0.05 level of significance, test if the mean scores of the three rows are equal

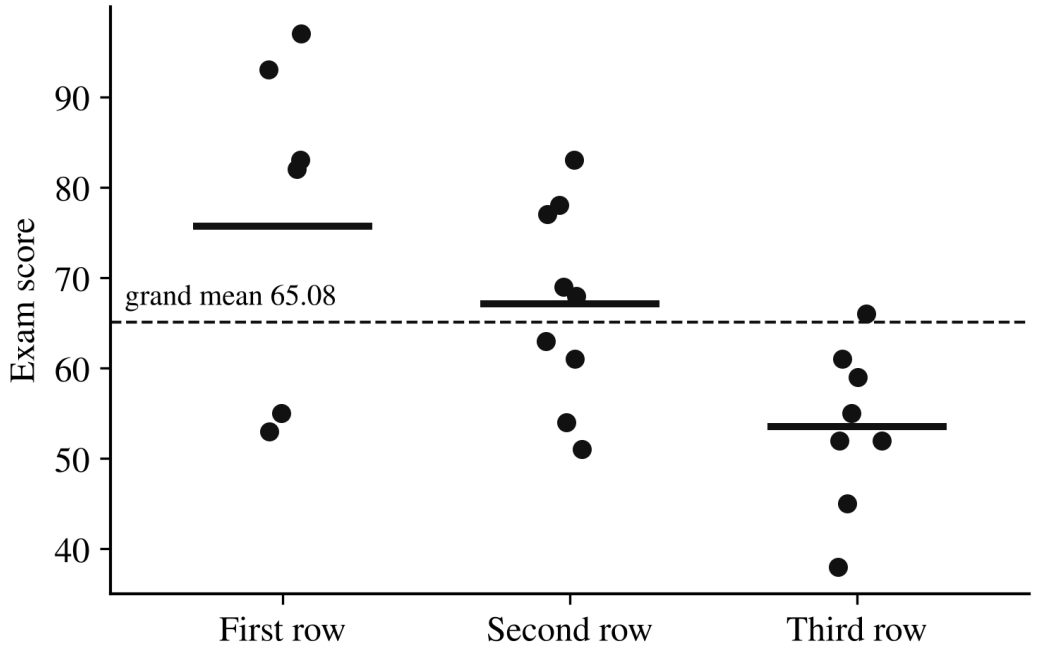
Example 2: Step 1 Hypotheses

1 Hypotheses

$$H_0: \ \mu_F = \mu_S = \mu_T$$

H_1: at least one mean is different

	First (F)	Second (S)	Third (T)
Sample Size (n_i)	7	9	8
Mean ($\bar{y}_{i.}$)	75.71	67.11	53.50
St. Dev. (S_i)	17.63	10.95	8.96
Variance S_i^2	310.90	119.86	80.29



Example 2: Step 2 Sums of Squares

Grand mean
Weighted mean

$$\bar{y}_{..} = \frac{7(75.71) + 9(67.11) + 8(53.50)}{7 + 9 + 8} = 65.08$$

SSA
Between rows

$$\text{SSA} = \sum_{i=1}^{k} n_i (\bar{y}_{i.} - \bar{y}_{..})^2$$

$$= 7(75.714 - 65.083)^2 + 9(67.111 - 65.083)^2 + 8(53.500 - 65.083)^2 = 1901.52$$

SSE
Within rows

$$\text{SSE} = \sum_{i=1}^{k} (n_i - 1) {S_i}^2 = 6(310.905) + 8(119.861) + 7(80.286) = 3386.32$$

SST
Total

$$\text{SST} = \text{SSA} + \text{SSE} = 1901.52 + 3386.32 = 5287.83$$

The formulas show the means rounded to 3 decimals. The results use the exact values

Example 2: Steps 3 and 4

3 Degrees of freedom

$$\mathrm{df}(A) = 3 - 1 = 2$$

$$\mathrm{df}(\mathrm{E}) = 24 - 3 = 21$$

$$\mathrm{df}(Total) = 24 - 1 = 23$$

4 Mean squares

$$\mathrm{MSA} = \frac{1901.52}{2} = 950.76$$

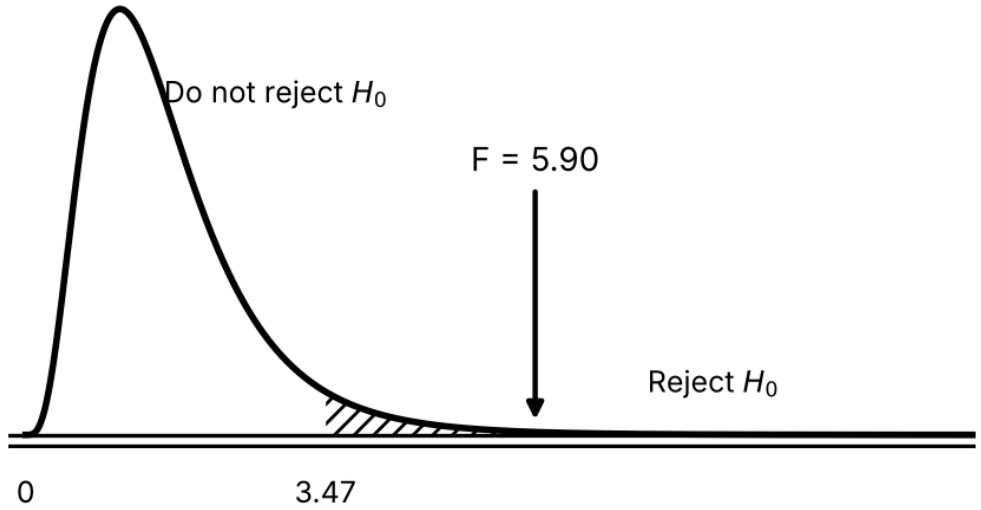
$$\mathrm{MSE} = \frac{3386.32}{21} = 161.25$$

Example 2: Step 5 F Test

5 F statistic

$$F = \frac{\text{MSA}}{\text{MSE}} = \frac{950.76}{161.25} = 5.90$$

- $\alpha = 0.05$, df = (2, 21)
- Critical value $F_{0.05}(2, 21) = 3.47$ from the F table
- The p-value is $P(F > 5.90) \approx 0.009$ from software



Decision

F = 5.90 > 3.47, so we **reject** H_0. The p-value 0.009 is less than 0.05

Example 2: Step 6 Decision

Source	SS	df	MS	F	Critical F
Between rows	1901.52	2	950.76	5.90	3.47
Within rows	3386.32	21	161.25		
Total	5287.83	23			

Decision

F = 5.90 is greater than 3.47 and the p-value 0.009 is less than 0.05. We reject H_0

Conclusion

There is enough evidence that the mean scores of the three rows are different. To find which rows differ we need confidence intervals or post hoc tests (not in this course)