Fourier Integral

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We have seen that a periodic function defined on a finite interval $(-L, L)$ or $(0, L)$ can be represented by a Fourier series which converges to the periodic extension of the function outside the interval. We can say that Fourier series are associated only with periodic functions. We now try to represent a given non periodic function defined either on an infinite interval $(-\infty, \infty)$ or semi-infinite interval $(0, \infty)$.

From Fourier Series to Fourier Integral

If f is a function defined on $\mathbb{R}$
If we let

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)dt,$$

and

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)dt,$$

then we have the definition

Definition

The Fourier integral of a function f defined on $(-\infty, \infty)$ is defined by

$$f(x) = \frac{1}{\pi}\int_0^{\infty} A(\lambda)\cos(\lambda x)d\lambda + \frac{1}{\pi}\int_0^{\infty} B(\lambda)\sin(\lambda x)d\lambda.$$

We now give convergence theorem for the Fourier integral.

Theorem

If f is absolutely integrable ($\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$), and f, f' are piecewise continuous on every finite interval, then the Fourier integral of f converges to $f(x)$ at a point of continuity and converges to $\frac{f(x^+)+f(x^-)}{2}$ at a point of discontinuity.

We now define the Fourier integral for even and odd functions as we have done with Fourier series.

Example

Express the function

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1, \end{cases}$$

as a Fourier integral. Hence evaluate $\displaystyle\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda$ and deduce the value of $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda$.

Solution.

Since

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda = \frac{1}{\pi}\int_{0}^{\infty}\int_{-1}^{1}\cos\lambda(t-x)dt \\ &= \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\lambda(t-x)}{\lambda}\Big|_{-1}^{1} d\lambda \\ &= \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\lambda(1-x)-\sin\lambda(-1-x)}{\lambda}d\lambda \\ &= \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\lambda(1+x)+\sin\lambda(1-x)}{\lambda}d\lambda \\ &= \frac{2}{\ }\int^{\infty}\frac{\sin\lambda\cos\lambda x}{\ }d\lambda \end{aligned}$$

then

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda = \begin{cases} \frac{\pi}{2}, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

At $x = \pm 1$, is discontinuous and the integral has the value $\frac{1}{2}(\frac{\pi}{2} + 0) = \frac{\pi}{4}$.
By setting $x = 0$, we get

$$\int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda = \frac{\pi}{2}.$$

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} 0, & -\infty < x < -\pi \\ -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & \pi < x < \infty. \end{cases}$$

Solution.
We have

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^{\infty}\int_{-\pi}^{0} -\cos\lambda(t-x)dtd\lambda + \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\cos\lambda(t-x)dtd\lambda \\ &\quad -\frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(t-x)}{\lambda}\Big|_{-\pi}^{0}d\lambda + \frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(t-x)}{\lambda}\Big|_{0}^{\pi}d\lambda \\ &= -\frac{1}{\pi}\int_0^{\infty}\frac{-\sin\lambda x + \sin\lambda(\pi+x)}{\lambda}d\lambda + \frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(\pi-x) +}{\lambda} \\ &= 2\int_0^{\infty}\frac{(1-\cos\lambda\pi)}{\lambda}\sin(\lambda x)d\lambda. \end{aligned}$$

This Fourier integral converges at the discontinuities points $-\pi, 0, \pi$ respectively to

$$\frac{f((-\pi)^+) + f((-\pi)^-)}{2} = \frac{-1}{2},$$

$$\frac{f(0^+) + f(0^-)}{2} = 0,$$

$$\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{1}{2}.$$

Fourier Sine and Cosine Integrals

When f is an odd function on the interval $(-\infty, \infty)$. then

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)dt = 0,$$

since $f(t)\cos(\lambda t)$ is odd. But

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)dt = 2\int_{0}^{\infty} f(t)\sin(\lambda t)dt,$$

since $f(t)\sin(\lambda t)$ is even. Consequently (**??**) becomes

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\sin(\lambda t)dt\sin(\lambda x)d\lambda \\ &= \frac{2}{\pi}\int_0^\infty C(\lambda)\sin(\lambda x)d\lambda, \end{aligned} \tag{1}$$

where

$$C(\lambda) = \int_0^\infty f(t)\sin(\lambda t)dt.$$

This is the Fourier sine integral for the function f.

When f is even function on the interval $(-\infty, \infty)$, then

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)dt = 2\int_{0}^{\infty} f(t)\cos(\lambda t)dt,$$

since $f(t)\cos(\lambda t)$ is even. But

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)dt = 0,$$

since $f(t)\sin(\lambda t)$ is odd. Consequently (**??**) becomes

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos(\lambda t)dt\cos(\lambda x)d\lambda \\ &= \frac{2}{\pi}\int_0^\infty D(\lambda)\cos(\lambda x)d\lambda. \end{aligned} \tag{2}$$

where

$$D(\lambda) = \int_0^\infty f(t)\cos(\lambda t)dt.$$

This is the Fourier cosine integral for the function f.

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} |\sin x|, & |x| \leq \pi, \\ 0, & |x| \geq \pi, \end{cases}$$

and deduce that

$$\int_0^\infty \frac{\cos \lambda\pi + 1}{1 - \lambda^2} \cos(\frac{\pi\lambda}{2}) d\lambda = \frac{\pi}{2}.$$

Solution. We observe that the function f is even on the interval $(-\infty, \infty)$. So It has a Fourier cosine integral given by (**??**), that is

$$f(x) = \frac{2}{\pi}\int_0^{\infty} D(\lambda)\cos(\lambda x)d\lambda,$$

where

$$\begin{aligned} D(\lambda) &= \int_0^{\infty} f(t)\cos(\lambda t)dt = \int_0^{\pi} \sin t \cos(\lambda t)dt \\ &= \int_0^{\pi} \frac{\sin t(1-\lambda) + \sin t(1+\lambda)}{2}dt \\ &= \frac{-\cos t(1-\lambda)}{2(1-\lambda)}\Big|_0^{\pi} - \frac{\cos t(1+\lambda)}{2(1+\lambda)}\Big|_0^{\pi} \end{aligned}$$

Thus

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{1}{1-\lambda^2} [\cos \pi\lambda + 1] \cos(\lambda x) d\lambda.$$

Since f is continuous on the whole interval $(-\infty, \infty)$, the above integral converges to the given function $f(x)$. Setting $x = \pi/2$ in (**??**), we get

$$\int_0^\infty \frac{1}{1-\lambda^2} [\cos \pi\lambda + 1] \cos(\frac{\lambda\pi}{2}) d\lambda = \frac{\pi}{2}.$$

The Complex Form of Fourier Integral

Since $\cos\lambda(t-x)$ is an even function of λ, it follows from (**??**) that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda.$$

Also since $\sin\lambda(t-x)$ is an odd function on λ, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\lambda(t-x)dtd\lambda = 0.$$

If we multiply (**??**) by i and add to (**??**), we obtain

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\left[\cos\lambda(t-x) + i\sin\lambda(t-x)\right]dtd\lambda \\ &= \frac{1}{2\pi}\int\int f(t)e^{i\lambda(t-x)}dtd\lambda \end{aligned}$$

Example

Find the complex form of the Fourier integral for the function

$$f(x) = \begin{cases} e^x, & |x| \leq 1 \\ 0, & |x| > 1. \end{cases}$$

Solution.
We have

$$\begin{aligned}\beta(\lambda) &= \int_{-\infty}^{\infty} f(t)e^{i\lambda t}dx. = \int_{-1}^{1} e^{(i\lambda+1)t}dx\\ &= \frac{1}{(i\lambda+1)}\left. e^{(i\lambda+1)t}\right|_{-1}^{1} = \frac{1}{(i\lambda+1)}\left(e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right)\\ &= \frac{1-i\lambda}{1+\lambda^2}\left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right].\end{aligned}$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-i\lambda}{1+\lambda^2}\left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right] e^{-i\lambda x}d\lambda.$$

Exercises

1. Find the Fourier integral for the following functions:
    1. $f(x) = \begin{cases} 0, & x < 0, \\ e^{-x}, & x > 0 \end{cases}$
    2. $g(x) = \begin{cases} 0, & -\infty < x < -2 \\ -2, & -2 < x < 0 \\ 2, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$
    3. $h(x) = e^{-|x|} \cos x,$
    4. $k(x) = e^{-|x|} \sin x$
    5. $M(x) = \begin{cases} 0, & -\infty < x < -1 \\ 2x, & -1 < x < 1 \\ 0, & 1 < x < \infty \end{cases}$
    6. $N(x) = \begin{cases} 0, & x < 0, \\ \sin x, & 0 \leq x < \pi \end{cases}$
    7. $f(x) = \begin{cases} x, & -\pi \leq x \leq \pi \\ 0, & |x| > \pi \end{cases}$
    8. $f(x) = \begin{cases} -1, & -\pi \leq x \leq 0, \\ 1, & 0 < x < \pi \end{cases}$