Fourier Series

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Orthogonal Functions

We will use a new tool called inner product to define orthogonal functions and sets of orthogonal functions.

Definition

The inner product of two continuous functions f and g on the interval $[\alpha, \beta]$ is the scalar (real number)

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x)dx.$$

Definition

We say that the two continuous functions f and g are orthogonal on the interval $[\alpha, \beta]$ if

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x)dx = 0.$$

Example

The two functions $f(x) = \cos x$ and $g(x) = \sin x$ are orthogonal on the interval $[-\pi, \pi]$ since

$$(f, g) = \int_{-\pi}^{\pi} \cos x . \sin x dx = 0.$$

Example

The two functions $f(x) = x$ and $g(x) = e^{|x|}$ are orthogonal on any symmetric interval $[-A, A]$, where A is a positive real constant. We can use the fact that f is odd and g is even, or by using integration by parts, it can be easily checked that

$$(f, g) = \int_{-A}^{A} x e^{|x|} dx = 0.$$

Definition

We say that the set of continuous functions

$$\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots, \varphi_n(x), \dots\},$$

is orthogonal on the interval $[\alpha, \beta]$ if

$$(\varphi_n, \varphi_m) = \int_\alpha^\beta \varphi_n(x)\varphi_m(x)dx = 0, \quad \forall\ n \neq m.$$

Definition

We define the norm of a function f on the interval $[\alpha, \beta]$ in terms of the inner product as the quantity

$$\|f\| = \sqrt{(f, f)} = \left(\int_{\alpha}^{\beta} f^2(x) dx \right)^{1/2}.$$

Definition

If $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is an orthogonal set of continuous functions on the interval $[\alpha, \beta]$ with the property that $\|\varphi_n\| = 1$ for $n = 1, 2, \ldots$,then the set $\{\varphi_n(x)\}_{n \geq 1}$ is said to be an orthonormal set on the interval $[\alpha, \beta]$.

Definition

A set of real-valued functions

$$\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\},$$

is said to be orthogonal with respect to a weight function $w(x) > 0$ on the interval $[\alpha, \beta]$ if

$$(\varphi_n, \varphi_m)_{w(x)} = \int_{\alpha}^{\beta} w(x)\varphi_n(x)\varphi_m(x)dx = 0, \quad \forall\ n \neq m.$$

Example

Show that the set of functions

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin mx, \cos mx, \dots\},$$

is orthogonal on the interval $[-\pi, \pi]$. Find the corresponding orthonormal set on $[-\pi, \pi]$.

Solution. We have to show that

$$\begin{aligned}(1, \sin nx) &= 0,\ (1, \cos nx) = 0,\ (\sin nx, \sin mx) = 0,\ (\cos\\ (\sin nx, \cos mx) &= 0,\ \ \forall n \neq m.\end{aligned}$$

$$\begin{aligned}
(1, \sin nx) &= \int_{-\pi}^{\pi} \sin nx dx = -\frac{1}{n} \cos nx|_{-\pi}^{\pi} = 0, \\
(1, \cos nx) &= \int_{-\pi}^{\pi} \cos nx dx = \frac{1}{n} \sin nx|_{-\pi}^{\pi} = 0, \\
(\sin nx, \sin mx) &= \int_{-\pi}^{\pi} \sin nx \sin mx dx \\
&= \int_{-\pi}^{\pi} \frac{\cos(n-m)x - \cos(n+m)x}{2} dx = 0, \;\; n \neq r
\end{aligned}$$

$$
\begin{aligned}
(\cos nx, \cos mx) &= \int_{-\pi}^{\pi} \cos nx \cos mx dx \\
&= \int_{-\pi}^{\pi} \frac{\cos(n-m)x + \cos(n+m)x}{2} dx = 0, \ \ n \neq \\
(\sin nx, \cos mx) &= \int_{-\pi}^{\pi} \sin nx \sin mx dx \\
&= \int_{-\pi}^{\pi} \frac{\sin(n-m)x + \sin(n+m)x}{2} dx = 0.
\end{aligned}
$$

To determine the orthonormal set on $[-\pi, \pi]$, we have to divide each element by its norm.

$$\|1\|^2 = \int_{-\pi}^{\pi} dx = 2\pi,$$

$$\begin{aligned}
\|\sin mx\|^2 &= \int_{-\pi}^{\pi} (\sin mx)^2 dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2mx}{2} dx = \pi, \\
\|\cos mx\|^2 &= \int_{-\pi}^{\pi} (\cos mx)^2 dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2mx}{2} dx = \pi.
\end{aligned}$$

Hence the orthonormal set on $[-\pi, \pi]$

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\sin mx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}}, \dots\right\}.$$

Example

Show that the set of functions $\left\{\cos\frac{n\pi x}{2}, \sin\frac{n\pi x}{2}\right\}_{n\geq 1}$ is orthogonal on the interval $[-2, 2]$. What would be the orthonormal set on $[-2, 2]$.

Solution. We have

$$\begin{aligned}
(\cos\frac{m\pi x}{2}, \sin\frac{n\pi x}{2}) &= \int_{-2}^{2} \cos\frac{m\pi x}{2}.\sin\frac{n\pi x}{2}dx \\
&= \int_{-2}^{2} \frac{\sin\frac{(n-m)\pi x}{2} + \sin\frac{(n+m)\pi x}{2}}{2}dx \\
&= 0, \quad n \neq m,
\end{aligned}$$

$$
\begin{aligned}
\left(\cos\frac{n\pi x}{2},\cos\frac{m\pi x}{2}\right) &= \int_{-2}^{2}\cos\frac{n\pi x}{2}.\cos\frac{m\pi x}{2}dx\\
&= \int_{-2}^{2}\frac{\cos\frac{(n-m)\pi x}{2}+\cos\frac{(n+m)\pi x}{2}}{2}dx\\
&= 0, \quad n\neq m,
\end{aligned}
$$

$$\begin{aligned}(\sin\frac{n\pi x}{2},\sin\frac{m\pi x}{2}) &= \int_{-2}^{2}\sin\frac{n\pi x}{2}.\sin\frac{m\pi x}{2}dx\\ &= \int_{-2}^{2}\frac{\cos\frac{(n-m)\pi x}{2}-\cos\frac{(n+m)\pi x}{2}}{2}dx\\ &= 0,\qquad n\neq m.\end{aligned}$$

So the given set of functions is orthogonal on $[-2,2]$.To find the orthonormal set, we have to compute

$$\left\|\cos\frac{m\pi x}{2}\right\|^2 = \int_{-2}^{2}(\cos\frac{m\pi x}{2})^2 dx = \int_{-2}^{2}\frac{1+\cos m\pi x}{2}dx = 2,$$

$$\left\|\sin\frac{m\pi x}{2}\right\|^2 = \int_{-\pi}^{\pi}(\sin\frac{m\pi x}{2})^2 dx = \int_{-2}^{2}\frac{1-\cos m\pi x}{2}dx = 2.$$

Hence the orthonormal set is

$$\left\{\frac{\cos\frac{n\pi x}{2}}{\sqrt{2}}, \frac{\sin\frac{n\pi x}{2}}{\sqrt{2}}\right\}_{n\geq 1}.$$

Example

Show that the functions

$$f(x) = 1, g(x) = 2x, h(x) = 4x^2 - 2,$$

are orthogonal with respect to the weight function $w(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$.

Solution.

$$(1, 2x)_{w(x)} = \int_{-\infty}^{\infty} 2xe^{-x^2} dx = -2 \int_{-\infty}^{\infty} d(e^{-x^2}) = -2 \; e^{-x^2}\Big|_{-\infty}^{\infty} = 0,$$

$$\begin{aligned}(1, 4x^2 - 2)_{w(x)} &= \int_{-\infty}^{\infty} (4x^2 - 2)e^{-x^2} dx = -\int_{-\infty}^{\infty} 2xd(e^{-x^2}) - 2 \\ &= -2xe^{-x^2}\Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} e^{-x^2} dx - 2\int_{-\infty}^{\infty} e^{-x^2} dx\end{aligned}$$

In the same way and by integration by parts, we find that

$$(2x, 4x^2 - 2)_{w(x)} = 0.$$

Trigonometric Series

Definition

A trigonometric series is a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where the coefficients a_n and b_n are constants. If the coefficients a_n and b_n satisfy certain conditions which will be specified later on, then the series is called Fourier series. almost all trigonometric series encountered in physical problems are of Fourier type.

Observe that each term in the sum (12) satisfies

$$\begin{aligned}\cos(x+2\pi) &= \cos x,\ \ \sin(x+2\pi)=\sin x,\ldots,\\ \cos n(x+2\pi) &= \cos nx,\ \ \sin n(x+2\pi)=\sin nx,\ldots\end{aligned}$$

Hence if the series (12) converges for all x in the domain of f, then its sum $f(x)$ must also satisfy the property

$$f(x+2\pi)=f(x).$$

A function f satisfying (26) is called periodic of period 2π. In general a function f such that

$$f(x+T)=f(x),\quad T\neq 0,\ (T>0).$$

for all x in the domain of f is said to be periodic with period T.

Periodic functions are of common occurrence in many physical and engineering problems; for example in conduction of heat and mechanical vibration. It is useful to express these functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

within a desired range of values of the variable x. Such a series is known as a Fourier Series as mentioned before.

Fourier Series

Suppose that a periodic function f has the trigonometric representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

In order to determine the coefficients a_n and b_n in terms of the function f, we need the following integral results

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0, & n \neq m \\ \pi, & m = n \neq 0, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \begin{cases} 0, & n \neq m \\ \pi, & m = n \neq 0, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0,$$

for all n and m.
First if we multiply both sides of Eq (3) by $\cos mx$ and integrate over the interval $(-\pi, \pi)$, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \cos mx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos nx \cos mx dx \\ &+ \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin nx \cos mx dx.\end{aligned}$$

If term by term integration of the series is allowed, then we obtain

$$\int_{-\pi}^{\pi} f(x)\cos mx dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx \cos mx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nx \cos mx dx. \quad (1)$$

If $m = 0$, then all terms on the right-hand side of Eq (1) are zero except the first one and we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

For any positive m, we use identities (3) and (30), and find that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx, \quad m = 1, 2, \ldots$$

To find b_m, we multiply $f(x)$ by $\sin mx$ and proceeding in the same way, we obtain

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx, \quad m = 1, 2, \dots$$

We define the Fourier series as the trigonometric series (12) in which the coefficients a_0, a_m and b_m are computed from a function $f(x)$ by the formulas (**??**), (**??**) and (**??**). The series (12) is then called the Fourier series of the function $f(x)$.

Theorem

Every uniformly convergent trigonometric series is a Fourier series. More precisely, if the series (12) *converges uniformly for all* $x \in (-\pi, \pi)$, *then* $f(x)$ *is continuous for all* x, $f(x)$ *has period* 2π *and* (12) *is the Fourier series of* $f(x)$.

Corollary

If two trigonometric series converge uniformly and have the same sum for all $x \in (-\pi, \pi)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{a'_0}{2} + \sum_{n=1}^{\infty}(a'_n \cos nx + b'_n \sin nx),$$

then the series are identical. That is $a_0 = a'_0$, $a_n = a'_n$, $b_n = b'_n$ *for* $n = 1, 2, \dots$
In particular if the series (12) *converges uniformly to zero for all* $x \in (-\pi, \pi)$, *the all coefficients are zero.*

Convergence of Fourier Series

A function f is said to be piecewise continuous on an interval $[a, b]$, if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, so that
1. f is continuous on each open interval (x_{i-1}, x_i).
2. $f(x_i^+) = \lim\limits_{x \to x_i^+} f(x)$ and $f(x_i^-) = \lim\limits_{x \to x_i^-} f(x)$, $i = 1, \ldots n-1$,

such that both limits exist.

Theorem

Suppose that f and f' are piecewise continuous on the interval $[-T, T]$. Further, suppose that f is defined outside the interval $[-T, T]$, so that it is periodic with period $2T$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}),$$

whose coefficients are given by

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}dx, \quad n = 1, 2, \dots,$$

Theorem

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}dx, \quad n = 1, 2, \dots,$$

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx.$$

The Fourier series converges to $f(x)$ at all points x, where f is continuous, and to$[f(x^+) + f(x^-)]/2$ at all points x where f is discontinuous. For $x = \pm T$, the series converges to $[f(-T)^+ + f(T)^-]/2$.

Remark 1 : To obtain a better understanding of the content of the theorem, it is helpful to consider some classes of functions that fail to satisfy the assumed conditions. Functions that are not included in the theorem are primarily those with infinite discontinuities in the interval $[-T, T]$, such as $\frac{1}{x-2}$ as $x \to 2$, or $\ln(x - T)$ as $x \to T^+$. Functions having an infinite number of jump discontinuities in this interval are also excluded.

Even and Odd Functions

Two symmetry properties of functions will be useful in the study of Fourier series. A function $f(x)$ that satisfies $f(-x) = f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the $y-$axis. This function is said to be even. For example

$$f(x) = \sqrt{2 + x^4}, g(x) = e^{-|x|}, \ x \in \mathbb{R}$$

$$h(x) = \cos x + \ln(1 + x^2), \ x \in \mathbb{R}$$

$$k(x) = \begin{cases} |\sin x|, & |x| \leq \pi, \\ 0, & |x| > \pi. \end{cases}$$

A function f that satisfies $f(-x) = -f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the origin. It is said to be an odd function. For example

$$f(x) = e^{|x|} \sin x, \; x \in \mathbb{R},$$

$$h(x) = \sqrt{1+x^2} \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$k(x) = \begin{cases} x-1, & 0 < x < 1, \\ x+1, & -1 < x < 0, \\ 0, & |x| > 1, \end{cases}$$

$$M(x) = x^{1/3} - \sin x, \; x \in \mathbb{R}.$$

Properties of Symmetric Functions

Theorem

If $f(x)$ is an even piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx.$$

If $f(x)$ is an odd piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)dx = 0.$$

Theorem

For an even function, we have the Fourier coefficients

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad n = 1, 2, \dots,$$
$$b_n = 0, \ n = 1, 2, \dots$$

For an odd function, we have the Fourier coefficients:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, \dots,$$
$$a_n = 0, \quad n = 0, 1, 2, \dots$$

Remark 2 : If we want to expand a function f on the interval $(0, L)$, with $f(x+L) = f(x)$ then we let $T = L/2$, that is $1/T = 2/L$, and $n\pi/T = 2n\pi/L$, then

$$
\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L f(x)dx, \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx, \quad n = 1, 2, \dots, \\
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx, \quad n = 1, 2, \dots
\end{aligned}
$$

Example

Assume that there is a Fourier series converging to the function

$$\begin{aligned} f(x) &= |x|, \quad |x| \leq T \\ f(x+2T) &= f(x). \end{aligned}$$

Compute the Fourier series for the given function.

Solution.

The Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right).$$

Since $f(-x) = f(x)\ \forall x \in [-T, T]$, then f is even on $[-T, T]$, hence $b_n = 0,\ n = 1, 2, \ldots$

We have

$$a_0 = \frac{2}{T} \int_0^T f(x)dx = T,$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}dx, \quad n = 1, 2, \ldots, \\ &= \frac{2}{T}\int_0^T x\cos\frac{n\pi x}{T}dx \\ &= \frac{2T}{(n\pi)^2}\left(\cos n\pi - 1\right), \quad n = 1, 2, \ldots \end{aligned}$$

Thus the Fourier series for the function f is given by

$$f(x) = \frac{T}{2} - \frac{4T}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{T}.$$

Observe that from the obtained Fourier series, we can deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This follows from the fact that the Fourier series converges to $f(0) = 0$ at $x = 0$.

Example

Find the Fourier series of the function

$$\begin{aligned} f(x) &= \frac{1}{2}\pi - |x| \\ f(x+2\pi) &= f(x) \text{ for all } x \in \mathbb{R}. \end{aligned}$$

Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Solution.

The interval $[-\pi, \pi]$ can be partitioned to give the two open intervals $(-\pi, 0)$ and $(0, \pi)$.

$$f(x) = \frac{1}{2}\pi - x, \quad \text{in } (0, \pi),$$

and

$$f'(x) = -1.$$

Clearly both f and f' are continuous and have limits as $x \to 0$ from the right, and as $x \to \pi$ from the left. The situation in $(-\pi, 0)$ is similar. Consequently f and f' are piecewise continuous on $[-\pi, \pi]$, so f satisfies the conditions of Theorem (15). We easily check that $f(-x) = f(x)\ \forall x \in [-\pi, \pi]$, that is f is even on $[-\pi, \pi]$ and thus

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx = 0,$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx dx \\ &= \frac{2}{n^2\pi}(1 - \cos n\pi), \quad n = 1, 2, \dots \end{aligned}$$

$$b_n = 0, \quad n = 1, 2, \dots$$

Hence

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x.$$

Since there are no jumps, one must expect convergence everywhere. Since $f(0) = \frac{1}{2}\pi$, it follows from the above Fourier series that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Example

Let

$$f(x) = \begin{cases} 0, & -3 < x \leq 0 \\ 3, & 0 < x < 3, \end{cases}$$

such that $f(x+6) = f(x)$ for all x. Find the Fourier series for this function and determine where it converges.

Solution.
The interval $[-3,3]$ can be partitioned to give the two open intervals $(-3,0)$ and $(0,3)$. In $(0,3)$, $f(x)=3$ and $f'(x)=0$. Clearly both f and f' are continuous and have limits as $x\to 0^+$, and $x\to 0^-$ which exist and different. Same situation in $(-3,0)$. Consequently f and f' are piecewise continuous on $[-3,3]$, so f satisfies the conditions of Theorem (15). We compute the Fourier coefficients to find that

$$a_0=\frac{1}{3}\int_0^3 3dx=3,$$

$$a_n = \int_0^3 \cos\frac{n\pi x}{3}dx = 0, \quad n \neq 0,$$

$$b_n = \int_0^3 \sin\frac{n\pi x}{3}dx = 0 = \frac{1}{n\pi}(1-\cos n\pi) = \begin{cases} 0, & n \text{ is even} \\ \frac{6}{n\pi}, & n \text{ is odd} \end{cases}$$

Thus

$$f(x) = \frac{3}{2} + \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{3}.$$

At the discontinuous points $x = 0, \pm 3$, we see from the above relation that the Fourier series converges to $\frac{3}{2}$. This is exactly the mean value of the limits from the right and the left. So we might define f at these points to have the value $\frac{3}{2}$.

Example

Obtain the Fourier series to represent the function

$$f(x) = \frac{1}{4}(\pi - x)^2, 0 < x < 2\pi,$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution.
Let

$$f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \ \ x \in (0, \pi).$$

We have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 dx = \frac{-1}{4\pi} \left.\frac{(\pi - x)^3}{3}\right|_0^{2\pi} = \frac{\pi^2}{6},$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx dx \\
&= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - \{-2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin \cdots}{n \cdots}\right.\right. \\
&= \frac{1}{4\pi}\left(\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right) = \frac{1}{n^2},
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \sin nx dx \\
&= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - \{-2(\pi - x)\}\left(-\frac{\sin nx}{n^2}\right) + 2 \right. \\
&= \frac{1}{4\pi}\left[\left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right)\right] = 0.
\end{aligned}
$$

Thus

$$f(x) = \frac{1}{4}(\pi - x)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

At $x = 0$, we have

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

From which it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example

Expand the function

$$f(x) = x \sin x, \quad 0 < x < 2\pi,$$

as a Fourier series

Solution.

Let

$$f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

We have

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x \sin x dx = \frac{1}{\pi} \left[x(-\cos x) - 1.(-\sin x) \right]_0^{2\pi} = -2,$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos nx)dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]dx. \end{aligned}$$

Integration by parts leads to

$$a_n = -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, \quad n \neq 1.$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x dx = \frac{2}{\pi}\int_0^{2\pi} x\sin 2x dx = -1/2.$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx \sin x)dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]dx.
\end{aligned}
$$

Integration by parts leads to

$$
b_n = \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0, \; n \neq
$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x dx = \frac{1}{2\pi}\int_0^{2\pi} x\left(1 - \cos 2x\right) dx = \pi.$$

Thus

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx \\ &= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1}\cos nx. \end{aligned}$$

Example

Expand the function

$$f(x) = x \sin x,$$

as a Fourier series in the interval $-\pi < x < \pi$, and deduce that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots = \frac{\pi - 2}{4}.$$

Solution. Let

$$f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Since $f(x) = x\sin x$ is even on $(-\pi, \pi)$, then $b_n = 0$, and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x dx = \frac{2}{\pi}\left[x(-\cos x) - 1.(-\sin x)\right]_0^{\pi} = 2,$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx dx = \frac{1}{\pi}\int_0^{\pi} x(2\sin x\cos nx)dx \\ &= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]dx. \end{aligned}$$

Integration by parts leads to

$$\begin{aligned} a_n &= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \\ &= \frac{(-1)^{n+1}}{n-1} - \frac{(-1)^{n+1}}{n+1} = \begin{cases} \frac{2}{n^2-1}, & n \text{ is odd} \\ \frac{-2}{n^2-1}, & n \text{ is even} \end{cases}, \quad n \neq 1. \end{aligned}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x dx = \frac{1}{\pi}\int_0^{2\pi} x \sin 2x dx = -1/2.$$

Therefore

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx.$$

Setting $x = \pi/2$, we obtain

$$\frac{\pi}{2} = 1 - 1/2 - 2\left(\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \dots\right).$$

From which it follows that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \dots = \frac{\pi - 2}{4}.$$

Example

Find a Fourier series to represent the function

$$f(x) = x - x^2, \; x \in [-\pi, \pi].$$

Deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution.
We write

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{-2}{3}\pi^2,$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx dx = -\frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx \\ &= \frac{4}{n^2}(-1)^{n+1}, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx \\ &= \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

Hence

$$x - x^2 = \frac{-1}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

By setting $x = 0$, we obtain

$$\frac{-1}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 0,$$

from which it follows that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Example

Expand in Fourier series the function

$$f(x) = 1 - \frac{2}{\pi}|x|, \; x \in [-\pi, \pi],$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Solution.
The interval $[-\pi, \pi]$ can be partitioned to give the two open intervals $(-\pi, 0)$ and $(0, \pi)$. In $(0, \pi)$,

$$f(x) = 1 - \frac{2x}{\pi},$$

and

$$f'(x) = -\frac{2}{\pi}.$$

Clearly both f and f' are continuous on $[-\pi, \pi]$. Consequently, there are piecewise continuous on $[-\pi, \pi]$, so f satisfies the conditions of Theorem (15). Now observe that when $-\pi \leq x \leq 0$, that is $0 \leq -x \leq \pi$, we have

$$f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x),$$

and when $0 \leq x \leq \pi$, that is $-\pi \leq -x \leq 0$, we have

$$f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x).$$

We conclude that f is an even function on the interval $[-\pi, \pi]$. So $b_n = 0$, and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Then

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right) dx = 0,$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx dx \\ &= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} \\ &= \frac{4}{\pi^2 n^2}\left[1 - (-1)^n\right] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8}{\pi^2 n^2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= \begin{cases} 1+\frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1-\frac{2x}{\pi}, & 0 \leq x \leq \pi, \end{cases} \\ &= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos nx \\ &= \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}. \end{aligned}$$

Setting $x=0$, we get

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

Fourier Cosine and Sine Series

Recalling that the Fourier series for an odd function defined on $[-L, L]$ consists only of sine terms, that is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

We can obtain the form (7) by extending the function $f(x)$, $0 < x < L$, to the interval $(-L, L)$ in such a way that the extended function is odd. This can done by defining the function

$$f_o(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0, \end{cases}$$

and extending $f_o(x)$ to all x using the $2L-$periodicity. The function $f_o(x)$ is odd, so its Fourier series contains only sine terms. $f_o(x)$ is called the odd $2L-$periodic extension of $f(x)$. The resulting Fourier series expansion is called a half-range expansion for $f(x)$.
In the same way, we can define the even $2L-$periodic extension of as the function

$$f_e(x) = \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L < x < 0, \end{cases}$$

with

$$f_e(x + 2L) = f_e(x).$$

Hence the definition

Definition

Let f be piecewise continuous on the interval $[0, L]$. The Fourier cosine series of f on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{nx\pi}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots$$

Definition

The Fourier sine series of $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin \frac{nx\pi}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots$$

Example

Compute the Fourier sine series of the function

$$f(x) = \cos\frac{\pi x}{3},\ 0 < x < 3.$$

Solution. We extend f as an odd function on $[-3, 3]$

$$f_o(x) = \begin{cases} \cos\frac{\pi x}{3}, & 0 \le x < 3, \\ -\cos\frac{\pi x}{3} & -3 \le x < 0. \end{cases}$$

The Fourier sine series representation of

$$f(x) = \cos\frac{\pi x}{3}$$

is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{nx\pi}{3}, \quad 0 < x < 3,$$

where

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 \cos\frac{\pi x}{3}\sin\frac{n\pi x}{3}dx = \frac{1}{3}\int_0^3 \Big(\sin\frac{(n+1)\pi x}{3} - \sin\frac{(n-1}{3} \\ &= \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even.} \end{cases} \end{aligned}$$

According to Fourier theorem, equality holds for $0 < x < 3$, but not at $x = 0$ and $x = 3$

$$\cos\frac{\pi x}{3} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)}\sin\frac{2nx\pi}{3}, \quad 0 < x < 3.$$

At $x = 0$ and $x = 3$, the Fourier series converges to

$$\frac{f(0^+) + f(0^-)}{2} = 0,$$

and

$$\frac{f(3^+) + f(3^-)}{2} = 0,$$

respectively.

Example

Compute the Fourier cosine series for the function

$$f(x) = e^{2x},\ 0 \leq x \leq 1.$$

and deduce that

$$\frac{2}{e^2 - 1} = \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2} \left[e^2(-1)^n - 1\right].$$

Solution.

We extend f as an even function on $[-1, 1]$

$$f_e(x) = \begin{cases} e^{2x}, & 0 \leq x < 1, \\ e^{-2x} & -1 < x \leq 0. \end{cases}$$

The Fourier cosine series representation of

$$f(x) = e^{2x},$$

is

$$f(x) = e^{2x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \quad 0 \leq x \leq 1,$$

where

$$a_0 = 2\int_0^1 e^{2x}dx = e^2 - 1,$$

$$\begin{aligned} a_n &= 2\int_0^1 e^{2x}\cos n\pi x dx = 2\left[\frac{1}{2}\, e^{2x}\cos n\pi x\Big|_0^1 + \frac{1}{2}n\pi\int_0^1 e^{2x}\sin \right. \\ &= e^2(-1)^n - 1 + n\pi\left[\frac{1}{2}\, n\pi e^{2x}\sin n\pi x\Big|_0^1 - \frac{1}{2}n\pi\int_0^1 e^{2x}\cos n\pi \right. \\ &= e^2(-1)^n - 1 - \frac{1}{2}n^2\pi^2\int_0^1 e^{2x}\cos n\pi x dx. \end{aligned}$$

Hence

$$a_n = \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right].$$

The Fourier series is then

$$e^{2x} = \frac{e^2-1}{2} + \sum_{n=1}^{\infty} \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right]\cos n\pi x, \quad 0 \le x \le 1.$$

At $x = 0$, we have

$$\frac{1-e^2}{2} = \sum_{n=1}^{\infty} \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right].$$

Example

Find the Fourier sine series for the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi/2, \\ \pi - x, & \pi/2 \leq x \leq \pi. \end{cases}$$

Solution.

We extend f as an odd function on $[-\pi, \pi]$

$$f_o(x) = \begin{cases} x, & 0 \leq x \leq \pi/2, \\ \pi - x, & \pi/2 \leq x \leq \pi, \\ x, & -\pi/2 \leq x \leq 0, \\ -\pi - x, & -\pi \leq x \leq -\pi/2. \end{cases}$$

The Fourier sine series representation of f is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad 0 \leq x \leq \pi,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin x dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x) \sin nx dx.$$

We let $u = nx$ and integrate by parts, we get

$$b_n = \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \begin{cases} 0, & n \text{ even} \\ \frac{4(-1)^{(n-1)/2}}{\pi n^2}, & n \text{ odd}. \end{cases}$$

Hence the Fourier series is

$$\begin{aligned} f(x) &= \begin{cases} x, & 0 \leq x \leq \pi/2, \\ \pi - x, & \pi/2 \leq x \leq \pi. \end{cases} \\ &= \frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + \dots\right) \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin(2n+1)x. \end{aligned}$$

At $x = \pi$ and $x = -\pi$, the Fourier series converges to 0.

Complex Form of a Fourier Series

We have seen that the Fourier series in the interval $(-T, T)$ of a function f is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T}).$$

From Euler's formula we have

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}.$$

Substitution of (8) in (8) leads to

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{\frac{in\pi x}{T}} + e^{-\frac{in\pi x}{T}}}{2} \right) + b_n \left(\frac{e^{\frac{in\pi x}{T}} - e^{-\frac{in\pi x}{T}}}{2i} \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{\frac{in\pi x}{T}} + \left(\frac{a_n + ib_n}{2} \right) e^{-\frac{in\pi x}{T}} \\ &= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{T}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{T}}, \end{aligned}$$

where

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad n \geq 1.$$

Hence the complex form of the Fourier series of f is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{T}},$$

where

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(x) e^{-\frac{in\pi x}{T}} dx.$$

Example

Obtain the complex form of the Fourier series for the function $f(x) = e^{\lambda x} \quad -\pi < x < \pi$, in the form

$$e^{\lambda x} = \frac{\sinh \lambda\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{\lambda + in}{\lambda^2 + n^2} e^{inx},$$

and deduce that

$$\frac{\pi}{\lambda \sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$

Solution.

We look for the coefficients c_n in the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda x}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(\lambda-in)x}dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(\lambda-in)\pi}-e^{-(\lambda-in)\pi}}{\lambda-in}\right] \\
&= \frac{1}{2\pi}\left[\frac{e^{\lambda\pi}(\cos n\pi - i\sin n\pi)-e^{-\lambda\pi}(\cos n\pi + i\sin n\pi)}{\lambda-in}\right] \\
&= \frac{1}{2\pi(\lambda-in)}\left(e^{\lambda\pi}-e^{-\lambda\pi}\right)\cos n\pi \\
&= \frac{1}{2\pi(\lambda-in)}(e^{\lambda\pi}-e^{-\lambda\pi})\cos n\pi \\
&= \frac{1}{\quad}
\end{aligned}$$

Substituting this found c_n in the series to get

$$f(x) = e^{\lambda x} = \frac{\sinh \lambda\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(\lambda + in)}{\lambda^2 + n^2} e^{inx}.$$

Now by setting $x = 0$ in (3), we obtain

$$\frac{\pi}{\sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{\lambda}{\lambda^2 + n^2} + i \frac{n}{\lambda^2 + n^2} \right).$$

By equating the real part, we have

$$\frac{\pi}{\lambda \sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$