Power Series and Analytic Functions

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Definition

A power series in $(x - x_0)$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots,$$

where the coefficients a_n are constants. We also say that the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, is centred at x_0.We say that the series (1) converges at the point $x = \alpha$ if the infinite series $\sum_{n=0}^{\infty} a_n(\alpha - x_0)^n$ converges; that is, the limit of the sequence of partial sums $(S_n(x))_{n\in\mathbb{N}}$ exists. In other words

$$\lim_{n\to\infty} S_n(x) = \sum_{n=0}^{\infty} a_n(\alpha - x_0)^n = S,$$

Theorem

For the series (1), there is a number R $(0 \leq R \leq \infty)$, called the radius of convergence, such that (1) converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$. If the series (1) converges for all x, then $R = \infty$, and if it converges only at x_0, then $R = 0$.

We should mention that the series (1), might or might not converge at the end points $x_0 - R$ and $x_0 + R$ of the interval of convergence. Moreover, at the interior points of the interval of convergence, the power series (1) converges absolutely in the sense that $\sum_{n=0}^{\infty} |a_n(x - x_0)^n|$ converges.

Theorem

(Ratio Test) Suppose that $a_n \neq 0$ for all n, if

$$\lim_{n\to\infty} |a_{n+1}/a_n| = L \quad (0 \leq L \leq \infty),$$

then the radius of convergence of (1) is $R = 1/L$, with $R = \infty$ if $L = 0$ and $R = 0$ if $L = \infty$.

Theorem

(Root test) Suppose that $a_n \neq 0$ *for all* n, *if*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L \quad (0 \leq L \leq \infty),$$

then the radius of convergence of (1) *is* $R = 1/L$, *with* $R = \infty$ *if* $L = 0$ *and* $R = 0$ *if* $L = \infty$.

Example

Determine the convergence set of the power series

$$\sum_{n=0}^{\infty} \frac{4}{n^3}(x-2)^n.$$

Solution.
We have

$$\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} \left| \frac{4}{(n+1)^3} \frac{n^3}{4} \right| = 1 = L.$$

The radius of convergence is $R = 1$. Hence the series converges absolutely for $|x-2| < 1$ and diverges when $|x-2| > 1$. At the end point $x = 1$, the series becomes $\sum_{n=0}^{\infty} \frac{4}{n^3}(-1)^n$, this is an alternating series which converges, since $\frac{4}{n^3}$ decreases and tend to zero as $n \to \infty$. At the other end point $x = 3$, the series becomes $\sum_{n=0}^{\infty} \frac{4}{n^3}$ which converges as a $p-$series with $p > 1$. Thus the series converges for $x \in [1, 3]$.

Differentiation and Integration of a Power Series

The power series (1) defines a function $f(x)$ that is

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

The domain of such a function is the interval of convergence of the power series. The functions f' and $\int f(x)dx$ can be found by term by term differentiation and integration, since f is continuous, differentiable and locally integrable on the interval of convergence $(x_0 - R, x_0 + R)$. We have

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2},$$

and hence

$$\int f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+1}/(n+1) \quad \forall x \in (x_0 - R, x_0 + R).$$

Remark 1 : If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$, for all $x \in (x_0 - R, x_0 + R)$, then $a_n = 0$ for all n.

Shifting the Index of Summation

Combining two or more summations as a single one often requires a re-indexing, that is a shift in the index of summation. We illustrate this fact by the following example.

Example

Write

$$\sum_{n=2}^{\infty} na_n (n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1},$$

as a power series whose general term involves x^n.

Solution.

It is necessary that both summation indices start with the same number and the common power of x should be x^n.

Step 1 In both series, we make the common power x^n, that is, we add 2 in any term inside the summation and substrate 2 from the index of summation in the first series, and substrate 1 from any term in the summation and add 1 in the index of summation in the second series, we obtain

$$\begin{aligned}&\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}\\ =\ &\sum_{n=0}^{\infty}(n+2)(n+1)\, a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n.\end{aligned}$$

Step 2. We let the index of summation starts by 1 in both series on the right-hand side of (24), that is we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n\\ =\;& 2a_2+\sum_{n=1}^{\infty}[a_{n+2}(n+2)(n+1)+a_{n-1}]x^n.\end{aligned}$$

Series Solution of Second Order Linear Equations

In this section, we present an effective method for solving many second order linear differential equations with variable coefficients by means of infinite series. We shall refer to this as the method of power series solutions. Second order linear differential equations appear frequently in applied mathematics, especially in the process of solving some of the classical partial differential equations in mathematical physics. The following are some of the most important second order linear differential equations with variable coefficients which occur in applications. It is common to refer to these equations by the name that appears to the left of the differential equation

$$y'' - xy = 0, \quad \text{Airy's equation},$$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \text{Bessel's equation of order } \nu,$$

$$(1 - x^2)y'' - xy' + \nu^2 y = 0, \quad \text{Chebychev's equation},$$

$$y'' - 2xy' + 2\nu y = 0, \quad \text{Hermite's equation},$$

$$xy'' + (1 - x)y' + \nu y = 0, \quad \text{Laguerre's equation},$$

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0, \quad \text{Legendre's equation}.$$

All these equations can be solved by the method of power series.

Ordinary Points and Singular Points

We consider a second order homogeneous linear differential equation with variable coefficients of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

Definition

A point x_0 is called an ordinary point of the differential equation (1) if the two functions $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ are analytic at x_0, that means that there exist two positive real constants R_1, R_2 such that

$$a_1(x)/a_2(x) = \sum_{n=0}^{\infty} A_n(x - x_0)^n, \quad \text{for } |x - x_0| < R_1,$$

$$a_0(x)/a_2(x) = \sum_{n=0}^{\infty} B_n(x - x_0)^n, \quad \text{for } |x - x_0| < R_2.$$

If at least one of the functions $a_1(x)/a_2(x)$, and $a_0(x)/a_2(x)$ is not analytic at x_0, then x_0 is called a singular point of the differential equation (1).

In most differential equations of the form (1) that occur in applications, the coefficients $a_0(x)$, $a_1(x)$,and $a_2(x)$ are polynomials. After canceling common factors, the rational functions $a_1(x)/a_2(x)$, and $a_0(x)/a_2(x)$ are analytic at every point except where the denominator vanishes. The points at which the denominator vanishes are singular points of the differential equation, and all other real numbers are ordinary points.

Definition

A point x_0 is called a regular singular point of the differential equation (1) if it is a singular and the two functions $(x - x_0)a_1(x)/a_2(x)$, and $(x - x_0)^2 a_0(x)/a_2(x)$ are analytic at the point x_0.
If at least one of the preceding functions is not analytic at x_0, then x_0 is called an irregular singular point of the differential equation (1).

Example

Locate the ordinary points, regular singular points, and irregular singular points of the differential equation

$$(x^4 - x^2)y'' + (2x + 1)y' + x^2(x + 1)y = 0.$$

Solution.
Here $a_2(x) = x^4 - x^2$, $a_1(x) = 2x + 1$, $a_0(x) = x^2\,(x+1)$, and so

$$a_1(x)/a_2(x) = \frac{2x+1}{x^4 - x^2} = \frac{2x+1}{x^2(x-1)(x+1)},$$

and

$$a_0(x)/a_2(x) = \frac{x^2\,(x+1)}{x^4 - x^2} = \frac{1}{x-1}.$$

It follows from *Eq* (24) and *Eq* (24) that every real number except $0, 1$ and -1 is an ordinary point for *Eq* (**??**). To see which of the singular points $0, 1$ and -1 is a regular singular point and which is an irregular singular point for *Eq* (**??**), we need to examine the two functions $(x - x_0)a_1(x)/a_2(x)$, and $(x - x_0)^2 a_0(x)/a_2(x)$ at the points $0, 1$ and -1. At $x_0 = 0$, we have

$$(x - x_0)a_1(x)/a_2(x) = \frac{2x+1}{x(x-1)(x+1)},$$

$$(x-x_0)a_1(x)/a_2(x) = \frac{2x+1}{x^2(x-1)},$$

and

$$(x-x_0)^2 a_0(x)/a_2(x) = \frac{(x+1)^2}{x-1}.$$

Since both of these functions are analytic at $x_0 = -1$, we conclude that $x_0 = -1$ is a regular singular point for the differential equation (9).

Example

Locate the ordinary points, regular singular points, and irregular singular points of the differential equation

$$(x-1)^2y'' - (x^2-x)y' + y = 0.$$

Solution.
Here the functions

$$a_1(x)/a_2(x) = -\frac{x(x-1)}{(x-1)^2} = \frac{-x}{x-1},$$

and

$$a_0(x)/a_2(x) = \frac{1}{(x-1)^2}$$

are analytic at any real number except at $x_0 = 1$, so every real number x is an ordinary point of Eq (10), except $x_0 = 1$ is a singular point. We now see whether $x_0 = 1$ is regular or irregular. We have$(x - x_0)a_1(x)/a_2(x) = -x$, and $(x - x_0)^2 a_0(x)/a_2(x) = 1$. These functions are analytic at $x_0 = 1$, so $x_0 = 1$ is a regular singular point.

Exercises

In the following exercise answer true or false

1. The point $x_0 = -1$ is a regular singular point for the differential equation
$$(1-x^2)y'' - 2xy' + 12y = 0.$$
2. The point $x_0 = 0$ is an ordinary point for the differential equation
$$xy'' + (1-x)y' + 2y = 0.$$
3. The point $x_0 = 0$ is a singular point for the differential equation
$$(1+x)y'' - 2y' + 2xy = 0.$$

Power Series Solution About an Ordinary Point

In this section, we show how to solve any second order homogeneous differential equation with variable coefficients of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

in some interval about an ordinary point x_0. The point x_0 is usually dictated by the specific problem at hand which requires that we find the solution of the differential equation (5) that satisfies the given initial conditions of the form

$$y(x_0) = y_0, \ y'(x_0) = y_1.$$

Let us recall that if the coefficients $a_2(x), a_1(x)$, and $a_0(x)$ are polynomials in x, then a point x_0 is an ordinary point of *Eq* (5). In general, x_0 is an ordinary point if $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ have power series expansions of the form (7) and (7) with radius of convergence R_1 and R_2 respectively. The functions (7) and (7) are continuous on the interval

Theorem

[Solution about an ordinary point]
If x_0 is an ordinary point of the differential equation (5), then the general solution of (5) has a power series expansion about x_0

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

with positive radius of convergence.
The coefficients a_n for $n = 2, 3, 4, \dots$ of the series (11) can be obtained in terms of a_0 and a_1 by direct substitution of (11) in the differential equation (5) and equating coefficients of the same powers. Finally if (11) is the solution of (5), (5), then $a_0 = y_0$ and $a_1 = y_1$.

Example

Find the general solution of the differential equation

$$y' - 2xy = 0$$

about the ordinary point $x_0 = 0$.

Solution.
It is clear that $x_0 = 0$ is an ordinary point since there are no finite singular points. The solution of Eq (12) is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

then Eq (12) becomes

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0.$$

We first make the same power of x as x^n in both series in (32) by letting $k = n - 1$ in the first series and $k = n + 1$ in the second one, we have

Example

Find the general solution of the differential equation

$$4y'' + y = 0,$$

about the ordinary point $x_0 = 0$.

Solution.

The functions

$$a_1(x)/a_2(x) = 0,$$

and

$$a_0(x)/a_2(x) = 1/4,$$

are analytic for all $x \in \mathbb{R}$, then every point $x_0 \in \mathbb{R}$ is an ordinary point for the differential equation (13). The solution of *Eq* (13) is of the form

$$\sum^{\infty}$$

Remark 2 : The solution (33) can be obtained by using the characteristic equation

$$4m^2 + 1 = 0,$$

then

$$m = \pm \frac{1}{2} i,$$

so

$$y = C_1 \cos(x/2) + C_2 \sin(x/2),$$

where C_1 and C_2 are arbitrary constants.

Example

Find the general solution of the differential equation

$$(1-x^2)y'' - 2xy' + 20y = 0,$$

about the ordinary point $x_0 = 0$.

Solution.
We have

$$a_1(x)/a_2(x) = \frac{-2x}{1-x^2} = -2\sum_{n=0}^{\infty} x^{2n+1} \quad \text{for } |x| < 1,$$

and

$$a_0(x)/a_2(x) = \frac{20}{1-x^2} = 20\sum_{n=0}^{\infty} x^{2n} \quad \text{for } |x| < 1.$$

So the solution of *Eq* (14) is of the form

Example

Find the general solution of the differential equation

$$y'' - 2(x-1)y' + 2y = 0,$$

about the ordinary point $x_0 = 1$.

Solution.
The two functions

$$a_1(x)/a_2(x) = -2(x-1),$$

and

$$a_0(x)/a_2(x) = 2,$$

are analytic for all $x \in \mathbb{R}$, so the solution of *Eq* (15) can have the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

for all $x \in \mathbb{R}$. To determine the coefficients a_n, we have

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1},$$

and

$$y'' = \sum n(n-1) a_n (x-1)^{n-2},$$

Example

Solve the initial value problem by the method of power series about the initial point $x_0 = 0$.

$$\begin{cases} (1-x^2)y'' - xy' + 4y = 0, \\ y(0) = 1, \ y'(0) = 0. \end{cases}$$

Solution.
The two functions

$$a_1(x)/a_2(x) = \frac{-x}{1-x^2} = -\sum_{n=0}^{\infty} x^{2n+1} \quad \text{for } |x| < 1,$$

and

$$a_0(x)/a_2(x) = \frac{4}{1-x^2} = 4\sum_{n=0}^{\infty} x^{2n} \quad \text{for } |x| < 1,$$

are analytic for all $|x| < 1$, then the solution of the differential equation in (16) is given by

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ for } |x| < 1.$$

Hence

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

Example

Compute the first four coefficients of the power series solution about the given initial point

$$\begin{cases} xy'' - 2(x+1)y' + 2y = 0, \\ y(3) = 2, \quad y'(3) = 0. \end{cases}$$

Solution.

We have

$$a_1(x)/a_2(x) = \frac{-2(x+1)}{x} = \frac{-2(x-3+4)}{x-3+3} = -\frac{2}{3}\frac{(x-3)+4}{(1+\frac{x-3}{3})}$$

$$= -\frac{2}{3}((x-3)+4)\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{3^n}$$

$$= -2\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^{n+1}}{3^{n+1}} - 8\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)}{3^{n+1}}$$

and

$$a_0(x)/a_2(x) = \frac{2}{x} = \frac{2}{3(1+\frac{x-3}{3})} = \frac{2}{3}\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{3^n} \text{ for all } |x$$

Then $a_0(x)/a_2(x)$, and $a_1(x)/a_2(x)$ are analytic for all $|x-3| < 3$. Hence the solution of the differential equation in (17) has the form

Example

Use power series method to solve the non-homogeneous differential equation

$$y'' - xy = 2 + 3x - 4x^2,$$

about the ordinary point $x_0 = 0$.

Solution.
It is clear that $x_0 = 0$ is an ordinary point of the differential equation

$$y'' - xy = 0,$$

so the solution of (18) is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

for all $x \in \mathbb{R}$. The differential equation then becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 2 + 3x - 4x^2.$$

If we let $k = n-2$ in the first sum and $k = n+1$ in the second sum in (37), then we obtain

$$\sum^{\infty}(k+1)(k+2)a_{k+2}x^k - \sum^{\infty} a_{k-1}x^k = 2 + 3x - 4x^2.$$

Example

Use power series method to solve the non-homogeneous differential equation

$$y'' - 4xy' - 4y = e^x,$$

about the ordinary point $x_0 = 0$.

Solution.

The solution of the differential equation (19) is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

for all $x \in \mathbb{R}$. Since

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

The differential equation (19) takes the form

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=1}^{\infty} n a_n x^n - 4 \sum_{n=0}^{\infty} a_n x^n = e^x.$$

Exercises

In exercises 1 through 9, locate the ordinary points, regular singular points, and irregular singular points of the given differential equation

1. $xy'' - (2x+1)y' + y = 0.$
2. $(1-x)y'' - y' + xy = 0.$
3. $x^3(1-x^2)y'' + (2x-3)y' + xy = 0.$
4. $(1-x)^4y'' - xy = 0.$
5. $2x^2y'' + (x-x^2)y' - y = 0.$
6. $x^2(x^2-9)y'' - (x^2-9)y' + xy = 0.$
7. $(x^4-16)y'' + 2y = 0.$
8. $x(x^2+1)^3y'' + y' - 8xy = 0.$
9. $(x^3-8)^3y'' - 2xy' + y = 0.$

   In exercises 10 through 13, verify that all singular points of the differential equation are regular singular points

10. $x^2y'' + xy' + (x^2-\nu^2)y = 0.$ (Bessel equation)
11. $(1-x^2)y'' - xy' + \nu^2y = 0.$ (Chebyshev equation)