Power Series and Analytic Functions

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Definition

A power series in $(x - x_0)$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots, \qquad (1)$$

where the coefficients a_n are constants. We also say that the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, is centred at x_0.We say that the series (1) converges at the point $x = \alpha$ if the infinite series $\sum_{n=0}^{\infty} a_n(\alpha - x_0)^n$ converges; that is, the limit of the sequence of partial sums $(S_n(x))_{n\in\mathbb{N}}$ exists. In other words

$$\lim_{n\to\infty} S_n(x) = \sum_{n=0}^{\infty} a_n(\alpha - x_0)^n = S,$$

Theorem

For the series (1)*, there is a number* R $(0 \leq R \leq \infty)$*, called the radius of convergence, such that* (1) *converges absolutely for* $|x - x_0| < R$ *and diverges for* $|x - x_0| > R$ *. If the series* (1) *converges for all* x*, then* $R = \infty$*, and if it converges only at* x_0*, then* $R = 0$.

We should mention that the series (1), might or might not converge at the end points $x_0 - R$ and $x_0 + R$ of the interval of convergence. Moreover, at the interior points of the interval of convergence, the power series (1) converges absolutely in the sense that $\sum_{n=0}^{\infty} |a_n(x - x_0)^n|$ converges.

Theorem

(Ratio Test) Suppose that $a_n \neq 0$ *for all* n, *if*

$$\lim_{n\to\infty} |a_{n+1}/a_n| = L \quad (0 \leq L \leq \infty), \tag{2}$$

then the radius of convergence of (1) *is* $R = 1/L$, *with* $R = \infty$ *if* $L = 0$ *and* $R = 0$ *if* $L = \infty$.

Theorem

(Root test) Suppose that $a_n \neq 0$ *for all* n, *if*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L \quad (0 \leq L \leq \infty),$$

then the radius of convergence of (1) *is* $R = 1/L$, *with* $R = \infty$ *if* $L = 0$ *and* $R = 0$ *if* $L = \infty$.

Example

Determine the convergence set of the power series

$$\sum_{n=0}^{\infty} \frac{4}{n^3}(x-2)^n.$$

Solution.
We have

$$\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} \left| \frac{4}{(n+1)^3} \frac{n^3}{4} \right| = 1 = L.$$

The radius of convergence is $R = 1$. Hence the series converges absolutely for $|x-2| < 1$ and diverges when $|x-2| > 1$. At the end point $x = 1$, the series becomes $\sum_{n=0}^{\infty} \frac{4}{n^3}(-1)^n$, this is an alternating series which converges, since $\frac{4}{n^3}$ decreases and tend to zero as $n \to \infty$. At the other end point $x = 3$, the series becomes $\sum_{n=0}^{\infty} \frac{4}{n^3}$ which converges as a $p-$series with $p > 1$. Thus the series converges for $x \in [1, 3]$.

Differentiation and Integration of a Power Series

The power series (1) defines a function $f(x)$ that is

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The domain of such a function is the interval of convergence of the power series. The functions f' and $\int f(x)dx$ can be found by term by term differentiation and integration, since f is continuous, differentiable and locally integrable on the interval of convergence $(x_0 - R, x_0 + R)$. We have

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \tag{3}$$

and hence

$$\int f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+1}/(n+1), \quad \forall x \in (x_0 - R, x_0 + R). \tag{4}$$

Remark 1 : If $\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$, for all $x \in (x_0 - R, x_0 + R)$, then $a_n = 0$ for all n.

Shifting the Index of Summation

Combining two or more summations as a single one often requires a re-indexing, that is a shift in the index of summation. We illustrate this fact by the following example.

Example

Write

$$\sum_{n=2}^{\infty} na_n (n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1},$$

as a power series whose general term involves x^n.

Solution.
It is necessary that both summation indices start with the same number and the common power of x should be x^n.
Step 1 In both series, we make the common power x^n, that is, we add 2 in any term inside the summation and substrate 2 from the index of summation in the first series, and substrate 1 from any term in the summation and add 1 in the index of summation in the second series, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n. \qquad (5)$$

Step 2. We let the index of summation starts by 1 in both series on the right-hand side of (10), that is we have

$$\begin{aligned} & \sum_{n=0}^{\infty}(n+2)(n+1)\, a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n \\ = \quad & 2a_2 + \sum_{n=1}^{\infty}[a_{n+2}(n+2)(n+1) + a_{n-1}]x^n. \end{aligned}$$

Series Solution of Second Order Linear Equations

In this section, we present an effective method for solving many second order linear differential equations with variable coefficients by means of infinite series. We shall refer to this as the method of power series solutions. Second order linear differential equations appear frequently in applied mathematics, especially in the process of solving some of the classical partial differential equations in mathematical physics. The following are some of the most important second order linear differential equations with variable coefficients which occur in applications. It is common to refer to these equations by the name that appears to the left of the differential equation

$$y'' - xy = 0, \quad \text{Airy's equation},$$
$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \text{Bessel's equation of order } \nu,$$
$$(1 - x^2)y'' - xy' + \nu^2 y = 0, \quad \text{Chebychev's equation},$$
$$y'' - 2xy' + 2\nu y = 0, \quad \text{Hermite's equation},$$
$$xy'' + (1 - x)y' + \nu y = 0, \quad \text{Laguerre's equation},$$
$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0, \quad \text{Legendre's equation}.$$

All these equations can be solved by the method of power series.

Ordinary Points and Singular Points

We consider a second order homogeneous linear differential equation with variable coefficients of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \tag{6}$$

Definition

A point x_0 is called an ordinary point of the differential equation (1) if the two functions $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ are analytic at x_0, that means that there exist two positive real constants R_1, R_2 such that

$$a_1(x)/a_2(x) = \sum_{n=0}^{\infty} A_n(x-x_0)^n, \quad \text{for } |x-x_0| < R_1, \tag{7}$$

$$a_0(x)/a_2(x) = \sum_{n=0}^{\infty} B_n(x-x_0)^n, \quad \text{for } |x-x_0| < R_2. \tag{8}$$

If at least one of the functions $a_1(x)/a_2(x)$, and $a_0(x)/a_2(x)$ is not analytic at x_0, then x_0 is called a singular point of the differential equation (1).

In most differential equations of the form (1) that occur in applications, the coefficients $a_0(x)$, $a_1(x)$,and $a_2(x)$ are polynomials. After canceling common factors, the rational functions $a_1(x)/a_2(x)$, and $a_0(x)/a_2(x)$ are analytic at every point except where the denominator vanishes. The points at which the denominator vanishes are singular points of the differential equation, and all other real numbers are ordinary points.

Definition

A point x_0 is called a regular singular point of the differential equation (1) if it is a singular and the two functions $(x - x_0)a_1(x)/a_2(x)$, and $(x - x_0)^2 a_0(x)/a_2(x)$ are analytic at the point x_0.
If at least one of the preceding functions is not analytic at x_0, then x_0 is called an irregular singular point of the differential equation (1).

Example

Locate the ordinary points, regular singular points, and irregular singular points of the differential equation

$$(x^4 - x^2)y'' + (2x + 1)y' + x^2(x + 1)y = 0. \tag{9}$$

Solution.

Here $a_2(x) = x^4 - x^2$, $a_1(x) = 2x + 1$, $a_0(x) = x^2\,(x+1)$, and so

$$a_1(x)/a_2(x) = \frac{2x+1}{x^4 - x^2} = \frac{2x+1}{x^2(x-1)(x+1)}, \qquad (10)$$

and

$$a_0(x)/a_2(x) = \frac{x^2\,(x+1)}{x^4 - x^2} = \frac{1}{x-1}. \qquad (11)$$

It follows from *Eq* (10) and *Eq* (11) that every real number except $0, 1$ and -1 is an ordinary point for *Eq* (**??**). To see which of the singular points $0, 1$ and -1 is a regular singular point and which is an irregular singular point for *Eq* (**??**), we need to examine the two functions $(x - x_0)a_1(x)/a_2(x)$, and $(x - x_0)^2 a_0(x)/a_2(x)$ at the points $0, 1$ and -1. At $x_0 = 0$, we have

$$(x - x_0)a_1(x)/a_2(x) = \frac{2x+1}{x(x-1)(x+1)},$$

and

$$(x - x_0)a_1(x)/a_2(x) = \frac{2x+1}{x^2(x-1)},$$

and

$$(x - x_0)^2 a_0(x)/a_2(x) = \frac{(x+1)^2}{x-1}.$$

Since both of these functions are analytic at $x_0 = -1$, we conclude that $x_0 = -1$ is a regular singular point for the differential equation (9).

Example

Locate the ordinary points, regular singular points, and irregular singular points of the differential equation

$$(x-1)^2 y'' - (x^2 - x)y' + y = 0. \tag{12}$$

Solution.
Here the functions

$$a_1(x)/a_2(x) = -\frac{x(x-1)}{(x-1)^2} = \frac{-x}{x-1}, \tag{13}$$

and

$$a_0(x)/a_2(x) = \frac{1}{(x-1)^2} \tag{14}$$

are analytic at any real number except at $x_0 = 1$, so every real number x is an ordinary point of *Eq* (12), except $x_0 = 1$ is a singular point. We now see whether $x_0 = 1$ is regular or irregular. We have$(x - x_0)a_1(x)/a_2(x) = -x$, and $(x - x_0)^2 a_0(x)/a_2(x) = 1$. These functions are analytic at $x_0 = 1$, so $x_0 = 1$ is a regular singular point.