# Linear Differential Equations of Higher Order

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# Existence Theorem and Fundamental Set of Solutions

## Definition

The general linear differential equation of order n is an equation that can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = R(x),$$

where $R$ and the coefficients $a_1$, $a_2$ , . . . , $a_n$ are functions of $x$ defined on an interval $I$. The equation (1) is called a homogeneous linear differential equation if the function $R(x)$ is zero for all $x \in I$. Suppose that the coefficients $a_1$, $a_2$ , . . . , $a_n$ and the function $R$ are continuous on an interval $I$ such that $a_n(x)$ is never zero on $I$, then the equation (1) is said to be normal on $I$. If $R$ is not equal to zero on $I$, the equation (1) is called non - homogeneous linear differential equation.

## Example

$$\frac{d^2y}{dt^2} + w^2y = 0 \quad \text{(undamped free vibration ).}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E_0\cos(wt) \qquad (\ LRC - \text{circuit}).$$

$$x^2y'' + xy' + \lambda^2y = 0 \qquad \text{( Bessel differential equation ).}$$

Now we suppose that $y_1$, $y_2$,..., $y_k$ are solutions of the homogeneous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0,$$

then for all for all $c_1, c_2, \ldots, c_k$ in $\mathbb{R}$

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_k y_k,$$

is also a solution of (6).

So we have the following theorem

**Theorem (Linear combination)**

*Any linear combination of solutions of a homogeneous linear differential equation is also a solution.*

Now we give the existence and uniqueness theorem for an initial value problem ($IVP$) for nth-order linear differential equation.

## Theorem (Existence Theorem)

*Given an nth-order linear differential equation*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = R(x).$$

*that is normal on an interval $I$. Suppose $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are n arbitrary real numbers. Then there exists a unique solution $y = y(x)$ of* (4)
*satisfying the initial conditions*

$$y(x_0) = y_0,\ \ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}.$$

## Example

Discuss the existence of unique solution of $(IVP)$

$$\begin{cases} (x^2+1)y'' + x^2y' + 5y = \cos(x) \\ y(3) = 2, \quad y'(3) = 1. \end{cases}$$

**Solution.**

The functions

$$a_2(x) = x^2 + 1, a_1(x) = x^2, a_0(x) = 5,$$

and

$$R(x) = \cos(x).$$

are continuous on $I = \mathbb{R} = (-\infty, +\infty)$, and $a_2(x) \neq 0$ for all $x \in \mathbb{R}$, the point $x_0 = 3 \in I$. Then Theorem (4) assures that the $IVP$ has a unique solution on $\mathbb{R}$.

## Example

Find an interval $I$ for which the initial values problem $(IVP)$

$$\begin{cases} x^2y'' + \frac{x}{\sqrt{2-x}}y' + \frac{2}{\sqrt{x}}y = 0, \\ y(1) = 0, \quad y'(1) = 1. \end{cases}.$$

has a unique solution around $x_0 = 1$.

**Solution.** The function $a_2(x) = x^2$, is continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ if $x > 0$ or $x < 0$. But $x_0 = 1 \in I_1 = (0, \infty)$. The function $a_1(x) = \frac{x}{\sqrt{2-x}}$, is continuous on $I_2 = (-\infty, 2)$ and the function

$a_0(x) = \dfrac{2}{\sqrt{x}}$, is continuous on $I_1 = (0, \infty)$. Then the $(IVP)$ has a unique solution on $I_1 \cap I_2 = (0, 2) = I$. We can take any interval $I_3 \subset (0, 2)$ such that $x_0 = 1 \in I_3$. So $I$ is that the largest interval for which the $(IVP)$ has a unique solution.

## Example

Find an interval $I$ for which the *IVP*

$$\begin{cases} (x-1)(x-3)y'' + xy' + y = x^2, \\ y(2) = 1, \quad y'(2) = 0. \end{cases}$$

has a unique solution about $x_0 = 2$.

**Solution.**
The functions

$$a_2(x) = (x-1)(x-3) \;\; a_1(x) = x \, a_0(x) = 1 \, R(x) = x^2,$$

are continuous on $\mathbb{R}$. But $a_2(x) \neq 0$ if $x \in (-\infty, 1)$ or $x \in (1, 3)$ or $x \in (3, \infty)$. As $x_0 = 2$ so we take $I = (1, 3)$. Then the *IVP* has a unique solution on $I = (1, 3)$

## Example

From Theorem (4), we deduce that the *IVP*

$$\begin{cases} 3y''' + 5y'' - y' + 7y = 0, \\ y(1) = 0, \; y'(1) = 0, \quad y''(1) = 0. \end{cases}$$

has a unique solution $y = 0$ on $\mathbb{R}$.

### Definition (Linearly Dependent Solutions)

Let $f_1, \ f_2, \dots, f_n$ be $n$ functions defined on an interval $I$. The functions $f_1, f_2, \dots, \ f_n$ are said to be linearly dependent on $I$ if there exist $n$ constants $c_1, c_2, \dots, c_n$ not all zero ( i.e. $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$ ) such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \quad \text{for all} \quad x \in I.$$

## Example

Prove that the functions

$$f_1(x) = x, \; f_2(x) = e^x, \; f_3(x) = xe^x,$$

and

$$f_4(x) = (2 - 3x)e^x,$$

are linearly dependent on $\mathbb{R}$.

**Solution.**

$$f_4(x) = (2 - 3x)e^x = 2e^x - 3xe^x = 2f_2(x) - 3f_3(x) + 0f_1(x),$$

hence

$$0f_1(x) + 2f_2(x) - 3f_3(x) - f_4(x) = 0, \text{ for all } \quad x \in \mathbb{R}.$$

So there exist $c_1 = 0$, $c_2 = 2$, $c_3 = -3$, and $c_4 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) = 0, \text{ for all } \quad x \in \mathbb{R}.$$

Then $f_1$, $f_2$, $f_3$ and $f_4$ are linearly dependent on $\mathbb{R}$.

## Example

Show that $f_1(x) = \cos(2x)$, $f_2(x) = 1$, $f_3(x) = \cos^2(x)$ are linearly dependent on $\mathbb{R}$.

**Solution.**
We know that

$$f_3(x) = \cos^2(x) = \frac{1 + \cos(2x)}{2} = \frac{1}{2}\ f_2(x) + \frac{1}{2}\ f_1(x),$$

for all $x \in \mathbb{R}$.

Then there exist $c_1 = c_2 = \frac{1}{2}$ and $c_3 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

So $f_1$, $f_2$, and $f_3$ are linearly dependent on $\mathbb{R}$.

## Example

Show that

$$f_1(x) = 1,\ f_2(x) = \sec^2(x) \text{ and } \ f_3(x) = \tan^2(x)$$

are linearly dependent on $\left(0, \frac{\pi}{2}\right)$.

**Solution.** We know that

$$f_2(x) = \sec^2(x) = 1 + \tan^2(x) = f_1(x) + f_3(x),$$

hence

$$f_1(x) - f_2(x) + f_3(x) = 0 \text{ for all } x \in \left(0, \frac{\pi}{2}\right).$$

So there exist $c_1 = c_3 = 1$ and $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all} \quad x \in \left(0, \frac{\pi}{2}\right).$$

So $f_1$, $f_2$ and $f_3$ are linearly dependent on $x \in \left(0, \frac{\pi}{2}\right)$.

## Definition (Linearly Independent Solutions)

Let $f_1, f_2, \ldots, f_n$ be n functions defined on an interval $I$. The functions $f_1, f_2, \ldots, f_n$ are said to be linearly independent on $I$ if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0, \text{ for all } \quad x \in I.$$

is true only for $c_1 = c_2 = \ldots = c_n = 0$.

## Example

Show that $f_1(x) = x$ and $f_2(x) = x^2$ are linearly independent on $I = [-1, 1]$.

**Solution.** Let $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } x \in I.$$

We have to prove that $c_1 = c_2 = 0$. As

$$c_1 x + c_2 x^2 = 0 \text{ for all } -1 \leq x \leq 1,$$

then for $x = 1$ and $x = -\frac{1}{2}$ we have

$$c_1 + c_2 = 0,$$

and

$$-\frac{1}{2} c_1 + \frac{1}{4} c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then $f_1$, and $f_2$ are linearly independent on $I$.

## Example

Show that

$$f_1(x) = \sin(x)\, f_2(x) = \sin(2x),$$

are linearly independent on $I = [0, \pi)$.

**Solution.** Let $c_1,\ c_2 \in I$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \ \text{ for all } x \in I.$$

We have to show that $c_1 = c_2 = 0$. In fact for $x = \frac{\pi}{4}$, and $x = \frac{\pi}{3}$ we have

$$\begin{cases} c_1 \sin(\frac{\pi}{4}) + c_2 \sin(\frac{\pi}{2}) = 0, \\ c_1 \sin(\frac{\pi}{3}) + c_2 \sin(2\frac{\pi}{3}) = 0, \end{cases}$$

hence

$$\frac{1}{\sqrt{2}}c_1 + c_2 = 0, \quad \frac{\sqrt{3}}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then $f_1$, and $f_2$ are linearly independent on $I$.

## Example

Show that

$$f_1(x) = 1, f_2(x) = e^x, \text{ and } \; f_3(x) = e^{-x}.$$

are linearly independent on $\mathbb{R}$.

**Solution.**
Let $c_1$, $c_2$, $c_3 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0, \quad \text{for all} \quad x \in \mathbb{R}.$$

We have to prove that $c_1 = c_2 = c_3 = 0$. In fact we have

$$c_1 + c_2 e^x + c_3 e^{-x} = 0, \quad \text{for all } x \in \mathbb{R},$$

then for the values $x = 0$, $x = 1$, $x = -1$, we have

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_{1+} c_2 e + c_3 e^{-1} = 0 \\ c_1 + c_2 e^{-1} + c_3 e = 0, \end{cases}$$

which implies that $c_1 = c_2 = c_3 = 0$. Then $f_1$, $f_2$ and $f_3$ are linearly independent on $\mathbb{R}$.

Now we shall obtain a sufficient condition that $n$ functions are linearly independent on an interval $I$. Let us assume that each of the functions $f_1, f_2, \dots, f_n$ is differentiable at least $(n-1)$ times in the interval $I$. Let $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0, \text{ for all } x \in I.$$

We have

$$\begin{cases} c_1 f_1'(x) + c_2 f_2'(x) + \ldots + c_n f_n'(x) = 0 \\ c_1 f_1''(x) + c_2 f_2''(x) + \ldots + c_n f_n''(x) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \ldots + c_n f_n^{(n-1)}(x) = 0, \end{cases}$$

for all $x \in I$. The nature of the solutions of these $n$ linear equations in $c_1, c_2, \ldots, c_n$ will be determined by the value of the determinant

$$W(x, f_1, f_2 \ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & \ldots & f_n'(x) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & \ldots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Now if $x_0 \in I$ such that $W(x_0, f_1, f_2, \dots, f_n) \neq 0$, then $c_1 = c_2 = \dots = c_n = 0$, and hence the functions $f_1, f_2, \dots, f_n$ are linearly independent on $I$.

**Definition**

The function $W(x, f_1, f_2, \ldots, f_n)$ defined by the equation (29) is called Wronskian of the functions $f_1, f_2, \ldots, f_n$.

## Example

Show that $f_1(x) = 1,\ f_2(x) = x,\ ,\dots,\ \ f_n(x) = x^{n-1}$ are linearly independent on $\mathbb{R}$.

**Solution.**
We calculate

$$W(x,\ f_1, f_2 \dots, f_n) = \begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 0 & 1 & 2x & \dots & (n-1)x^{n-2} \\ 0 & 0 & 2 & \dots & (n-1)(n-2)x^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (n-1)! \end{vmatrix}$$

and we find $W(x,\ f_1,\ f_2,\ \dots,\ f_n) = 0!1!2!\dots(n-1)! \neq 0$ for all $x \in \mathbb{R}$. Then $f_1,\ f_2,\ \dots,\ f_n$ are linearly independent on $\mathbb{R}$.

## Example

Prove that $f_1(x) = x^2,\ f_2(x) = x^2\ln(x)$ are linearly independent on $(0,\ \infty)$.

**Solution.**

We use the definition of

$$\begin{aligned} W(x,\ f_1, f_2) &= \begin{vmatrix} x^2 & x^2\ln(x) \\ 2x & 2x\ln(x) + x \end{vmatrix} \\ &= 2x^3\ln(x) + x^3 - 2x^3\ln(x) = x^3 \neq 0 \text{ for all } x \in ( \end{aligned}$$

then $f_1$ and $f_2$ are linearly independent on $(0,\ \infty)$.

**Example**

Show that

$$f_1(x) = x^2 \text{and } f_2(x) = x\,|x|\,,$$

are

($i$) linearly dependent on [0, 1]

($ii$) linearly independent on $[-1, 1]$

**Solution.**

$(i)$ on [0,1] we have

$$f_1(x) = f_2(x) = x^2,$$

hence

$$f_1(x) - f_2(x) = 0, \text{ for all } 0 \leq x \leq 1.$$

So there exist $c_1 = 1$, $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \text{ for all } 0 \leq x \leq 1.$$

Then $f_1$ and $f_2$ are linearly dependent on [0, 1].

$(ii)$ Let $c_1, c_2 \in \mathbb{R}$ be such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } -1 \leq x \leq 1,$$

hence

$$c_1 x^2 + c_2 x\,|x| = 0 \text{ for all } -1 \leq x \leq 1.$$

Now for $x = 1$ and $x = -1$ we have $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ which implies that $c_1 = c_2 = 0$. Then $f_1$ and $f_2$ are linearly independent on $[-1, 1]$.

**Remark 1 :**

**(i)** If $f_1, f_2, \ldots, f_n$ are linearly dependent on an interval $I$ and each of the functions $f_1, f_2, \ldots, f_n$ is differentiable at least $(n-1)$ times on $I$, then

$$W(x, f_1, f_2, \ldots, f_n) = 0, \text{ for all } x \in I.$$

For example, it was proved that

$$f_1(x) = 1, \; f_2(x) = \sec^2(x), \text{ and } \; f_3(x) = \tan^2(x).$$

are linearly dependent on $\left(0, \frac{\pi}{2}\right)$, then

$$\begin{aligned} & W(x, f_1, f_2, f_3) \\ = & \begin{vmatrix} 1 & \sec^2(x) & \tan^2(x) \\ 0 & 2\sec^2(x)\tan(x) & 2\tan(x)\sec^2(x) \\ 0 & 4\sec^2(x)\tan^2(x) + 2\sec^4(x) & 4\sec^2(x)\tan^2(x) + 2\sec \end{vmatrix} \\ = & \; 0, \end{aligned}$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

$(ii)$ If $W(x, f_1, f_2, \ldots, f_n) = 0$ for all $x \in I$, then the functions $f_1, f_2, \ldots, f_n$ may be linearly independent or dependent on $I$.

## Example

We consider the functions

$$f_1(x) = x^2 \text{ and } f_2(x) = x\,|x|\,.$$

on the interval $I = [-1, 1]$. Prove that

$$W(x, f_1, f_2) = 0, \quad \text{for all } x \in I.$$

**Solution.**

1. For $0 < x \leq 1$, we have

$$W(x,\ f_1, f_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0.$$

2. For $-1 \leq x < 0$, we have

$$W(x,\ f_1, f_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0.$$

3. For $x = 0$ we have

$$W(0, f_1(0), f_2(0)) = \begin{vmatrix} f_1(0) & f_2(0 \\ f_1'(0) & f_2'(0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

So $W(x, f_1, f_2) = 0$ for all $x \in [-1, 1]$, even these functions $f_1$ and $f_2$ are linearly independent on $[-1, 1]$ (see the example (13) ), where $f_2'(0) = 0$.

The main result in this section is given by the following theorem.

**Theorem**

*If* $y_1, y_2, \ldots, y_n$ *are solutions of the differential equation*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0,$$

*where each* $a_i(x)$ *is defined and continuous on an interval* $I$ *and* $a_n(x) \neq 0$ *for all* $x \in I$*, then* $y_1, y_2, \ldots, y_n$ *are linearly independent on* $I$ *if and only if*

$$W(x, \; y_1 \, y_2, \ldots, y_n) \neq 0 \quad \textit{for all} \quad x \in I.$$

## Example

We know that the functions $x$ and $x^2$ are linearly independent on the interval $-1 \leq x \leq 1$. However

$$W(x,\ f_1(x),\ f_2(x)) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2,$$

so that

$$W(0,\ f_1(0), f_2\ (0)) = 0,\ \text{where}\ \ x = 0 \in I = [-1, -1].$$

This fact does not contradict Theorem (22), because there is no second- order linear differential equation with the interval of definition $-1 \leq x \leq 1$ that has $x$ and $x^2$ as solutions. We can verify that $y_1 = x$ and $y_2 = x^2$ are solutions of the second-order linear differential equation

$$x^2 y'' - 2xy' + 2y = 0,$$

where the interval of definition $I$ must exclude $x = 0$, since we have assumed that $a_2(x) = x^2 \neq 0$ in $I$. So that we conclude that the Theorem (4) is not contradicted by this example, and we should distinguish between the functions which are linearly independent on an interval $I$ as algebraic functions, and the functions which are linearly independent on an interval $I$, and are solutions of a linear differential equation.

### Example

It is easy to see that the functions

$$y_1 = x, \;\; y_2 = x^2,$$

and

$$y_3 = x^3.$$

are solutions of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0.$$

Show that $y_1$, $y_2$ and $y_3$ are linearly independent on $(0, \infty)$.

**Solution.**

Here we have $a_3(x) = x^3 \neq 0$ for all $x > 0$ or $x < 0$. By using the Wronskian we have

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \neq 0.$$

for all $x \in (0, \infty)$, or for all $x \in (-\infty, 0)$. So $y_1$, $y_2$ and $y_3$ are linearly independent on $(0, \infty)$ or on $(-\infty, 0)$. But as algebraic functions $y_1$, $y_2$ and $y_3$ are linearly independent on $\mathbb{R}$.

**Definition (Fundamental Set of Solutions)**

Any set $y_1, y_2, \ldots, y_n$ of $n$ functions linearly independent solutions of the homogeneous linear $n$th-order differential equation (22) on an interval $I$ is said to be a fundamental set of solutions on $I$.

Here the number of functions which form the fundamental set of solutions on $i$ equals to the order of the equation (22).

### Theorem

*Let $y_1$, $y_2$, . . . , $y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (22) on an interval $I$. Then for any solution $y$ of Eq (22) on $I$, there exist $n$ constants $c_1$, $c_2$,. . . . , $c_n \in \mathbb{R}$, such that*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x).$$

**Theorem (Existence of a fundamental set)**

*There exist a fundamental set of solutions for homogeneous linear nth-order differential equation* (22) *on an interval* $I$.

## Definition (General Solution of the Homogeneous Equation)

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of homogeneous linear $n$th-order differential equation (22) on an interval $I$.The general solution of the equation (22) on $I$ is defined by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x), \quad x \in I,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. The general solution of (22) is also called the complete solution of (22).

## Example

Verify that $y_1 = e^{2x}$, and $y_2 = e^{-3x}$ form a fundamental set of solutions of the differential equation

$$y'' + y' - 6y = 0,$$

and find the general solution.

**Solution.**
Substituting

$$y_1 = e^{2x}, y_1' = 2e^{2x},\ y_1'' = 4e^{2x},$$

in the differential equation, we get

$$4e^{2x} + 2e^{2x} - 6e^{2x} = 0.$$

Hence $y_1 = e^{2x}$, is a solution of the differential equation. By the same method we can prove that $y_2 = e^{-3x}$, is also a solution of the differential equation. We now have

$$W(x, e^{2x}, e^{-3x}) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -5e^{-x} \neq 0 \text{ for all } x \in \mathbb{R}.$$

Then $y_1$ and $y_2$ are linearly independent on $\mathbb{R}$. From Theorem (**??**), we deduce the general solution of the differential equation is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

where $c_1, \ c_2 \in \mathbb{R}$.

### Example

It is easy to see that the functions

$$y_1 = e^x, \ y_2 = e^{2x}, \text{ and } y_3 = e^{3x},$$

are solutions of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

Find the general solution of the differential equation.

**Solution.**
Since

$$W(x, e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0,$$

for all $x \in \mathbb{R}$.

We deduce that

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

is the general solution of the differential equation.

## Example

Prove that

$$y_1 = x^3 e^x, \text{ and } y_2 = e^x,$$

are solutions of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0,$$

where $x > 0$. Find also the general solution of the differential equation.

**Solution.**

Substituting

$$y_1 = x^3 e^x,\ y_1' = 3x^2 e^x + x^3 e^x,\ y_1'' = 6xe^x + 6x^2 e^x + x^3 e^x,$$

in the differential equation we obtain

$$W(x, x^3e^x, e^x) = \begin{vmatrix} x^3e^x & e^x \\ 3x^2e^x + x^3e^x & e^x \end{vmatrix} = -3x^2e^x \neq 0, \text{ for all } \; x$$

Then

$$y_1 = x^3e^x,$$

and

$$y_2 = e^x.$$

are linearly independent on $(0, \infty)$, and we conclude that

$$y_c = c_1x^3e^x + c_2e^x,$$

is the general solution of the differential equation.

**Remark 2 :**
The property of general solution exists only in the homogeneous linear nth -order differential equation (22) but does not exist in the homogeneous non- linear differential equation, for example the differential equation

$$(xy' + 1)(yy' + 1) = 0.$$

is a non-linear first order differential equation has not general solution, because it has two family of curves of solutions $y = -\ln |xc_1|$ such that $x \neq 0$, and an arbitrary constant $c_1 \neq 0$, $y^2 + 2x = c_2$ where $y \neq 0$ and $c_2$ is an arbitrary constant.

## Example

Given that

$$y = c_1 e^x + c_2 e^{-x},$$

is a two parameters family of solutions of

$$y'' - y = 0 \text{ on } (-\infty, \infty),$$

find a curve of the family satisfying the initial conditions $y(0) = 0, \ y'(0) = 1$.

**Solution.**

From Theorem (4) the initial value problem

$$\begin{cases} y''(x) - y(x) = 0 \\ y(0) = 0 \ \ y'(0) = 1, \end{cases}$$

has a unique solution.

For $y(0) = 0$ we have $c_1 + c_2 = 0$ and for $y'(0) = 1$ we have $c_1 - c_2 = 1$, hence $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. So the unique solution of the initial value problem is

$$y = \frac{1}{2}(e^x - e^{-x}) = \sinh(x).$$