# First Order Differential Equations

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Introduction In this chapter we study several elementary methods for solving first -order differential equations.
Consider the equation of order one

$$F(x, y, y') = 0.$$

We suppose that the equation (1), with some conditions, can be written as

$$y' = \frac{dy}{dx} = f(x, y).$$

The equation (1) can be also written in the form

$$M(x, y)dx + N(x, y)dy = 0,$$

where $M$ and $N$ are two functions of $x$ and $y$.

# Initial-Value Problems

We are often interested in problems in which we seek a solution $y(x)$ of differential equation so that it satisfies prescribed side conditions. that is conditions imposed on the unknown $y(x)$ or its derivatives. On some interval $I$ containing $x_0$, the problem

$$\begin{cases} \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \\ y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}, \end{cases}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary specified real constants, is called an **initial value problem** ($IVP$). The values $y(x)$ and its first $n-1$ derivatives at a single point $x_0$ : $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions.**

Special cases: First and second-order ($IVPs$)

$$\begin{cases} \frac{dy}{dx} = f(x, y), \\ y(x_0) = y_0, \end{cases}$$

$$\begin{cases} \frac{d^2y}{dx^2} = f(x, y, y'), \\ y(x_0) = y_0,\ y'(x_0) = y_1, \end{cases}$$

are first and second-order initial value problems, respectively.

# Example



What is the function you know from calculus, that is equal to its derivative?

**Solution.** It is clear that $y = ce^x$ is one parameter family of solution of the simple first -order equation $y' = y$. All solutions in this family are defined on the interval $(-\infty, \infty)$ . If we impose an initial condition, say $y(0) = 4$, then substituting $x = 0$ and $y = 4$ in the family determines the constant $c = 4$. Thus $y = 4e^x$ is a solution of the $(IVP)$

$$\begin{cases} y' = y, \\ y(0) = 4. \end{cases}$$

Now if we demand that a solution curve pass through the point $(1, -3)$ rather than $(0, 4)$, then $y(1) = -3$ will yield $-3 = ce$ or $c = -3e^{-1}$. In this case we have $y = -3e^{x-1}$ is the solution of the $(IVP)$

$$\begin{cases} y' = y, \\ y(1) = -3. \end{cases}$$

# Example

It is easy to see that a one-parameter family of solutions of the first-order differential equation

$$y' + 2xy^2 = 0,$$

is

$$y = \frac{1}{x^2 + c}.$$

If we impose the initial condition $y(0) = -1$, then substituting $x = 0$, and $y = -1$ into the family of solutions gives $c = -1$. Thus

$$y = \frac{1}{x^2 - 1}.$$

**Remark 1 :** The domain of the function

$$y = \frac{1}{x^2 - 1},$$

is $R = \{x \in \mathbb{R},\ \ x \neq \pm 1\}$ .Then

$$R = \{x \in \mathbb{R},\ \ x > 1\} \cup \{x \in \mathbb{R},\ \ -1 < x < 1\} \cup \{x \in \mathbb{R},\ \ x < -1\}$$

But $x_0 = 0$ then $x_0 \in R_1 = \{x \in \mathbb{R}\ \ -1 < x < 1\}$. So the largest interval on which $y = \frac{1}{x^2-1}$ is a solution satisfying the condition $y(0) = -1$ is $-1 < x < 1$.
This example illustrates that the interval $I = (-1, 1)$ of definition of solution $y(x)$ depends on the initial condition $y(0) = -1$.

It is desirable to know in advance, when solving an initial value problem, whether its solution exists, and is it unique?. Now we state here without proof a straightforward theorem that gives conditions that are sufficient to guarantee the existence and uniqueness of solution of a first-order initial-value problem of the form

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

That is solve the equation $y' = f(x, y)$ subject to the initial condition $y(x_0) = y_0$.

# Existence Theorem

## Theorem

*Consider the differential equation of order one*

$$\frac{dy}{dx} = f(x, y).$$

*Let* $T = \{(x, y), \quad |x - x_0| \leq a, \ \ |y - y_0| \leq b,\}$, *be the rectangular region with the center* $(x_0, y_0)$. *Suppose that* $f$ *and* $\frac{\partial f}{\partial y}$ *are continuous functions of* $x$ *and* $y$ *on* $T$. *Under the conditions imposed on* $f(x, y)$ *above, an interval exists about* $x_0$, $|x - x_0| \leq h$, *and a function* $y(x)$ *which has the following properties*

## Theorem

1. $y = y(x)$ *is a solution of the equation* $(10)$ *on the interval* $|x - x_0| \leq h$.
2. $|y(x) - y_0| \leq b$ *on the interval* $|x - x_0| \leq h$.
3. $y = y(x_0) = y_0$.
4. $y$ *is the unique solution of the differential equation on the interval* $|x - x_0| \leq h$ *with* $y(x_0) = y_0$.

# Example

Find the largest region of the $xy-$plane for which the initial value problem

$$\begin{cases} \sqrt{x^2-4}\,y' = 1+\sin(x)\ \ln y, \\ y(3) = 4, \end{cases}$$

has a unique solution.

**Solution.**

$$y' = \frac{1+\sin(x)\ \ln y}{\sqrt{x^2-4}} = f(x,y).$$

$$y' = \frac{1}{\sqrt{x^2-4}} + \frac{\sin x}{\sqrt{x^2-4}} \ln y, \quad y > 0 \ \text{ and } \quad |x| > 2,$$

$$\frac{\partial f}{\partial y} = \frac{\sin x}{\sqrt{x^2-4}}\frac{1}{y}.$$

Then $f$ and $\frac{\partial f}{\partial y}$ are continuous on

$$\begin{aligned} R &= \{(x,y) \in \mathbb{R}^2, \ |x| > 2, \ y > 0\} \\ &= \{(x,y), x > 2, \quad y > 0\} \cup \{(x,y), x < -2, \quad y > 0\}. \end{aligned}$$

But the point $(3,4) \in R_1 = \{(x,y),\ x > 2,\quad y > 0\}$, then the largest region in $xy$-plane for which the $IVP$ has a unique solution is $R_1$. If we take any rectangular $R_2$ with center $(3,4)$ such that $R_2 \subset R_1$, then the $IVP$ has also a unique solution, but $R_2$ is not the largest region.

# Example



Determine the largest region for which the following initial value problem admits a unique solution.

$$\begin{cases} \ln(x-2)\frac{dy}{dx} = \sqrt{y-2}, \\ y(\frac{5}{2}) = 4. \end{cases}$$

# Example



Find the largest region of the $xy$- plane for which the following initial value problem has a unique solution

$$\begin{cases} \sqrt{\frac{x}{y}}y' = \cos(x+y), \quad y \neq 0, \\ y(1) = 1. \end{cases}.$$

**Solution.**
We have

$$y' = \cos(x+y)(\frac{x}{y})^{\frac{-1}{2}} = f(x,y).$$

Then

$$\frac{\partial f}{\partial y} = -\sin(x+y)(\frac{x}{y})^{\frac{-1}{2}} - \frac{1}{2}\cos(x+y)(\frac{x}{y})^{\frac{-3}{2}}(\frac{-x}{y^2}).$$

So $f$ and $\frac{\partial f}{\partial y}$ are continuous on $R = \left\{(x, y), \quad \dfrac{x}{y} > 0\right\}$, or

$R = \{(x, y), \ x < 0 \quad \text{and } y < 0\} \cup \{(x, y), \ x > 0 \ \text{ and } \ y > 0\}$.

But

$$(1, 1) \in R_1 = \{(x, y), \ x > 0, \ y > 0\}.$$

Then the largest region for which the given $(IVP)$ has a unique solution is $R_1$.

# Exercises



1. Determine and sketch the largest region of the $xy-$plane for which the following initial value problems have a unique solution
$$\begin{cases} \frac{dy}{dx} = \frac{y+2x}{y-2x}, \\ \quad y(1) = 0. \end{cases}$$

In problems 2- 10, determine a region of the $xy$-plane for which the given differential equations would have a unique solution whose graph passes through a point $(x_0, y_0)$ in the region.

2. $\frac{dy}{dx} = y^{\frac{2}{3}}$.
3. $\frac{dy}{dx} = \sqrt{xy}$.
4. $x\ \frac{dy}{dx} = y^{\frac{1}{3}}$.
5. $\frac{dy}{dx} - \ln y = \sqrt{x}$.
6. $(4 - y^2)y' = x^2 y$.
7. $\ln(x-1)y' = \sin^{-1}(y)$.
8. $(x^2 + y^2)y' = \sqrt{y}\ x$.

9. $(y - x)y' = y + x^2$.
10. $(1 + y^3)y' = \tan^{-1}(x)$.

In problems 11-14 determine whether Theorem (1) guarantees that the differential equation

$$y' = \sqrt{y^2 - 9}.$$

possesses a unique solution through the given point.

1. $(1, 4)$.
2. $(5, 3)$.
3. $(2, -3)$.
4. $(-1, 1)$.

# Separable Equations

We begin our study of methods for solving first -order differential equation by studying an equation of the form

$$M(x, y)dx + N(x, y)dy = 0,$$

where $M$ and $N$ are two functions of $x$ and $y$. Some equations of this type are so simple that they can be written in the form

$$F(x)dx + G(y)dy = 0.$$

that is, the variables can be separated. The solution can be written immediately. For, it is only a matter of finding a function $H$ such that

$$dH(x, y) = F(x)dx + G(y)dy = 0.$$

the solution of (2) is $H(x, y) = c$ where $c$ is an arbitrary constant.

# Example



Find the solution of differential equation

$$2x(y^2+y)dx+(x^2-1)ydy=0, \quad y\neq 0.$$

**Solution.**

The variables of the equation of (23) can be separated as

$$\frac{2x}{x^2-1}dx=\frac{-1}{y+1}dy, \quad x\neq \pm 1, \text{ and } \ y\neq -1,$$

by integrating two sides we have

$$\ln\left|x^2-1\right|+\ln\left|y+1\right|=c,$$

or

$$\ln\left|(x^2-1)(y+1)\right|=c.$$

What happens when $x = \pm 1$ and when, $y = 0$ or $y = -1$.
Going back to the original equation (23) we see that four lines $x = \pm 1$, $y = 0$ and $y = -1$ also satisfy the differential equation (23).
If we relax the restriction $c_1 \neq 0$, the curve $y = -1$ will be contained in the formula

$$y = -1 + \frac{c_1}{x^2 - 1} \text{ for } c_1 = 0.$$

However the curves $x = \pm 1$ and $y = 0$ are not contained in the same formula, for any values of $c_1$. Sometimes such curves are called *singular solutions* and the one parameter family of solutions

$$y = -1 + \frac{c_1}{x^2 - 1},$$

where $c_1$ is an arbitrary constant, is called the general solution.

# Example

Find the solution of the differential equation

$$(xy + x)dx = (x^2y^2 + x^2 + y^2 + 1)dy.$$

**Solution.**
We have

$$x(y + 1)dx = (x^2 + 1)(y^2 + 1)dy,$$

hence

$$\frac{xdx}{x^2 + 1} = \frac{y^2 + 1}{y + 1}dy, \quad y \neq -1,$$

then

$$\frac{xdx}{x^2 + 1} = \left[(y - 1) + \frac{2}{y + 1}\right] dy,$$

by integrating the two sides, we obtain

$$\ln(x^2+1)-(y-1)^2-\ln(y+1)^4=c.$$

So the family of curves (27) defines implicitly the solution of (26). We also see that $y=-1$ satisfies the equation (23) but it is not in the family (27), then $y=-1$ is a singular solution of (26).

# Example



Solve the initial value problem

$$\begin{cases} e^y \frac{dy}{dx} = \cos(2x) + 2e^y \sin^2(x) - 1, \\ y(\frac{\pi}{2}) = \ln 2. \end{cases}$$

**Solution.**
By separating the variables we have

$$\begin{aligned} e^y \frac{dy}{dx} &= 2e^y \sin^2(x) + \cos(2x) - 1, \\ &= e^y(1 - \cos(2x)) - (1 - \cos(2x)) \\ &= (e^y - 1)(1 - \cos(2x)), \end{aligned}$$

hence

$$\int \frac{e^y}{e^y - 1} dy = \int (1 - \cos(2x)) dx.$$

Consequently

$$\ln|e^y - 1| + \frac{\sin(2x)}{2} - x = c,$$

which is the solution of the differential equation. Now we use the initial condition

$$x = \frac{\pi}{2}, \quad y = \ln 2 \quad \Longrightarrow \quad \ln 1 + \frac{\sin \pi}{2} - \frac{\pi}{2} = c \implies c = -\frac{\pi}{2},$$

then the solution of initial value problem is

$$\ln|e^y - 1| + \frac{\sin 2x}{2} + \frac{\pi}{2} = 0.$$

# Equations with Homogeneous Coefficients

## Definition

Let $f$ be a function of $x$ and $y$ with domain $D$. The function $f$ is called homogeneous of degree $k \in \mathbb{R}$ if

$$f(tx, ty) = t^k f(x, y) \ \forall\ t > 0, \quad \text{and } \forall\ (x, y) \in D \ \text{ such that } (tx, t$$

# Example

1. It is easy to see that if $M(x,y)$ and $N(x,y)$ are both homogeneous and of the same degree, then the function $\frac{M(x,y)}{N(x,y)}$ is homogeneous of degree zero. We can take as an example the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2},$$

is homogeneous of degree zero.

2. The function

$$f(x,y) = x - 2y + \sqrt{x^2 + 4y^2},$$

is homogeneous of degree one.

For

$$\begin{aligned} f(tx, ty) &= tx - 2ty + \sqrt{(tx)^2 + 4(ty)^2} \\ &= |t| \left[ x - 2y + \sqrt{x^2 + 4y^2} \right], \\ &= tf(x, y). \end{aligned}$$

3. The function $f(x, y) = x \ln x - x \ln y$, is homogeneous of degree one because $f(x, y) = x \ln(\frac{x}{y})$, and

$$f(tx, ty) = (tx) \ln(\frac{tx}{ty}) = t \left[ x \ln(\frac{x}{y}) \right] = tf(x, y).$$

4. The functions

$$f(x,y) = x^2 + y^2 + \frac{x-y}{x+y},$$

and

$$f(x,y) = 2x - 3y + e^{x-y},$$

are not homogeneous.

We now consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0,$$

where $M$ and $N$ where homogeneous functions of the same degree. To find the solution of the equation (4) we put $u = \frac{y}{x}$, $x \neq 0$ or $u = \frac{x}{y}$, $y \neq 0$. Then the differential equation transforms to another equation with separable variables that we can solve by the method of section (2.2).

# Example

Solve the differential equation

$$(x^2 - xy + y^2)dx - xydy = 0.$$

**Solution.**
The coefficients in (34) are both homogeneous and of degree two in $x$ and $y$. Let $u = \frac{y}{x}$, $x \neq 0$, then

$$y = ux \qquad \Longrightarrow \quad dy = udx + xdu,$$

and we have

$$(x^2 - x^2u + x^2u^2)dx - x^2u(udx + xdu) = 0.$$

We divide this equation by $x^2$ to obtain

$$(1 - u + u^2)dx - u(udx + xdu) = 0,$$

or

$$(1 - u)dx - xudu = 0.$$

Hence we separate the variables to get

$$\frac{dx}{x} + \frac{udu}{u-1} = 0, \quad u \neq 1,$$

or

$$\frac{dx}{x} + \left[1 + \frac{1}{u-1}\right] du = 0,$$

a family of solutions is seen to be

$$\ln|x| + u + \ln|u-1| = \ln|c|, \;\; c \neq 0.$$

or

$$x(u-1)e^{u} = c_1, \quad x \neq 0, \;\; u \neq 1 \text{ and } c_1 \neq 0.$$

In terms of the original variables, these solutions are given by

$$x(\frac{y}{x} - 1)\exp(\frac{y}{x}) = c_1,$$

or

$$(y - x)\exp(\frac{y}{x}) = c_1, \quad x \neq 0 \quad \text{and} \quad y \neq x.$$

We see that $y = x$ is also is solution of the equation (34) and $y = x$ satisfies (36) for $c_1 = 0$. Then the family of solutions of the DE (34) is given by

$$(y - x)\exp(\frac{y}{x}) = c_1, \quad x \neq 0 \quad \text{and} \quad c_1 \in \mathbb{R}.$$

# Example



Solve the differential equation

$$\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0, \quad x \neq 0 \text{ and } y \neq -x.$$

# Example

Solve the initial value problem

$$ydx + x\left(\ln\frac{x}{y} - 1\right)dy = 0, \quad y(1) = e.$$

**Solution.**

The coefficients of the differential equation are homogeneous with degree one. So we can put $u = \frac{x}{y}$ then

$$x = yu \implies dx = ydu + udy,$$

# Example

Find the solution of the differential equation

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}, \; x > 0.$$

# Solving Some Differential Equations by Using Appropriate Substitution



If we have a differential equation of the form

$$\frac{dy}{dx} = f(Ax + By).$$

We substitute

$$u = Ax + By,$$

then

$$\frac{du}{dx} = A + B\frac{dy}{dx}.$$

# Example

Find the solution of the differential equation

$$\frac{dy}{dx} = (-2x + y)^2 - 7.$$

**Solution.**
Let

$$u = -2x + y,$$

then

$$u' = -2 + \frac{dy}{dx},$$

and

$$\frac{dy}{dx} = u' + 2 = u^2 - 7,$$

or

$$\frac{du}{dx} = u^2 - 9 \quad \Longrightarrow \frac{1}{6}\int \frac{1}{u-3}du - \frac{1}{6}\int \frac{1}{u+3}du = dx, \quad u \neq \pm 3,$$

so

$$\ln\left|\frac{u-3}{u+3}\right| - 6x = c,$$

then the solutions of the differential equation (41) is given by

$$\ln\left|\frac{-2x+y-3}{-2x+y+3}\right| - 6x = c, \quad \text{where } c \text{ is an arbitrary constant.}$$

# Example

Solve the differential equation by using an appropriate substitution

$$\frac{dy}{dx} = \frac{1-4x-4y}{x+y}, \;\; x+y \neq 0.$$

**Solution.**

We see that the two straight lines $1-4x-4y=0$, and $x+y=0$ are parallel, in this case we put $u=x+y$, hence $y' = u' - 1$, and we have $\frac{dy}{dx} = \frac{1-4u}{u} = \frac{du}{dx} - 1$. Or $\frac{du}{dx} = \frac{1-3u}{u}, \quad \Longrightarrow \quad \frac{u}{1-3u} du = dx, \quad u \neq 0$ and $1-3u \neq 0$.

Consequently

$$\frac{-1}{3}\int\left(1-\frac{1}{1-3u}\right)du=\int dx,$$

$$+\frac{u}{3}+\frac{1}{9}\ln|1-3u|+x=c,$$

then the solutions of the differential equation (43) is given by

$$\frac{x+y}{3}+\frac{1}{9}\ln|1-3x-3y|+x=c,$$

where $c$ is an arbitrary constant.

# Example

Solve the differential equation by using an appropriate substitution

$$\frac{dy}{dx} = \frac{x-y-3}{x+y-1}, \;\; x+y-1 \neq 0.$$

**Solution.**
We see that the two straight lines $x-y-3=0$, and $x+y-1=0$, are not parallel, in this case we find the point of intersection which is $(2,-1)$ and we put
$x-2=u, \quad y+1=v$. Or

$$x = u+2, \; y = v-1, \implies \quad dx = du, \quad dy = dv,$$

then $\frac{dv}{du} = \frac{u+2-(v-1)-3}{u+2+(v-1)-1} = \frac{u-v}{u+v}$.

So, we have the homogeneous differential equation

$$\frac{dv}{du} = \frac{u-v}{u+v}.$$

Hence we put $\frac{v}{u} = t$, where $u \neq 0$, then $v = ut$, and

$$\frac{dv}{du} = t + u\frac{dt}{du}.$$

So we deduce that

$$u\frac{dt}{du} = \frac{1-t}{1+t} - t = \frac{1-2t-t^2}{1+t}.$$

Or

$$\int \frac{du}{u} = \int \frac{1+t}{1-2t-t^2}dt, \;\; 1-2t-t^2 \neq 0,$$

$$\ln|u| + \frac{1}{2}\ln\left|1 - 2t - t^2\right| = c,$$

$$\ln\left[u^2\left|1 - 2\frac{v}{u} - \frac{v^2}{u^2}\right|\right] = 2c,$$

$$u^2 - 2vu - v^2 = c_1, \quad c_1 = \pm e^{2c}.$$

Then the solution of the differential equation (45) is given by

$(x-2)^2 - 2(x-2)(y+1) - (y+1)^2 = c_1$, where $c_1 \neq 0$ is an arbitrary

# Example

Solve the differential equation by using an appropriate substitution

$$\frac{dy}{dx} = \frac{y(1+xy)}{x(1-xy)}, \; x > 0, \quad y > 0 \quad \text{and} \quad xy \neq 1.$$

**Solution.**

We can solve this differential equation by using the substitution $u = xy$ or $y = \frac{u}{x}$ then

$$x\frac{dy}{dx} + y = \frac{du}{dx},$$

hence

$$x\frac{dy}{dx} = \frac{y(1+xy)}{(1-xy)}$$

$$\frac{du}{dx} - y = \frac{y(1+xy)}{(1-xy)}$$

$$\frac{du}{dx} - \frac{u}{x} = \frac{u}{x}(\frac{1+u}{1-u})$$

$$\frac{du}{dx} = \frac{2u}{x(1-u)}.$$

By separating the variables we have

$$\frac{1}{2}\int (\frac{1}{u} - 1)du = \int \frac{dx}{x},$$

$$\ln u - u - \ln x^2 = c \implies \frac{u}{x^2} = e^u c_1, \quad c_1 = e^c,$$

then the solution of the differential equation (48) is given by

$$\frac{y}{x} = e^{xy} c_1, \text{ where } c_1 \neq 0 \text{ is an arbitrary constant.}$$

# Exercises



In exercises 1 through 11, obtain a family of solutions

1. $3(3x^2 + y^2)dx - 2xydy = 0.$
2. $(x - y)dx + (2x + y)dy = 0.$
3. $x^2y' = 4x^2 + 7xy + 2y^2.$
4. $(x - y)(4x + y)dx + x(5x - y)dy = 0.$
5. $x(x^2 + y^2)(ydx - xdy) + y^6dy = 0.$

6. $\left[x\csc\left(\frac{y}{x}\right) - y\right]dx + xdy = 0.$
7. $xdx + \sin^2\left(\frac{y}{x}\right)\left[ydx - xdy\right] = 0.$
8. $(x - y\ln y + y\ln x)dx + x(\ln y - \ln x)dy = 0.$
9. $\frac{dy}{dx} = \frac{x+3y}{3x+y}.$
10. $-ydx + (x + \sqrt{xy})dy = 0.$
11. $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0.$

In exercises 12 through 18, find the solution of the initial value problem ($IVP$)

12 $$\begin{cases} (x-y)dx + (3x+y)dy = 0, \\ y(3) = -2. \end{cases}$$

13 $$\begin{cases} (y - \sqrt{x^2+y^2})dx - xdy = 0, \\ y(0) = 1. \end{cases}$$

14 $$\begin{cases} \left[x\cos^2(\frac{y}{x}) - y\right] dx + xdy = 0, \\ y(1) = \frac{\pi}{4}. \end{cases}$$

15 $$\begin{cases} y^2dx + (x^2 + 3xy + 4y^2)dy = 0, \\ y(2) = 1. \end{cases}$$

16 $$\begin{cases} y(x^2 + y^2)dx + x(3x^2 - 5y^2)dy = 0, \\ y(2) = 1. \end{cases}$$

17 $$\begin{cases} (x + ye^{\frac{y}{x}})dx - xe^{\frac{y}{x}}dy = 0, \\ y(1) = 0. \end{cases}$$

18 $\begin{cases} (x^2+2y^2)\frac{dx}{dy} = xy, \\ y(-1) = 1. \end{cases}$

19 Prove that by using the substitution $y = ux$, you can solve any equation of the form

$$y^n f(x)dx + H(x,y)(ydx - xdy) = 0,$$

where $H(x,y)$ is homogeneous in $x$ and $y$.

20 If $F$ is homogeneous of degree $k$ in $x$ and $y$, $F$ can be written in the form

$$F = x^k \varphi(\frac{y}{x}), \quad x > 0,$$

where $\varphi$ is a function can be determined from $F$.
In exercises 23 through 31, solve the given differential equation by using an appropriate substitution.

21 $\frac{dy}{dx} = (x + y + 1)^2$.

22 $\frac{dy}{dx} = \tan^2(x + y)$.

23 $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$.

24 $\frac{dy}{dx} = 1 + e^{y-x+5}$.

25 $\frac{dy}{dx} = \frac{1-x-y}{x+y}$.

26 $(x + 2y - 4)dx - (2x + y - 5)dy = 0$.

27 $(2x + 3y - 1)dx + (2x + 3y + 2)dy = 0$.

28 $x\,\frac{dy}{dx} = y\ln(xy)$.

29 $\frac{dy}{dx} = \frac{2y}{x} + \cos^2(\frac{y}{x^2}),\ x \neq 0$. (Hint put $u = \frac{y}{x^2}$).

A differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0,$$

is called *exact* if there is a function $F$ of $x$ and $y$ such that

$$dF(x,y) = M(x,y)dx + N(x,y)dy = 0.$$

Recall that the total differential of a function $F$ of $x$ and $y$ is given by

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

provided that the partial derivatives of the function $F$ with respect to $x$ and $y$ exist. If *Eq* (5) is exact, then (because of (5) and (5)) it is equivalent to

$$dF = 0.$$

Thus, the function $F$ is constant and the solution of the differential equation (5) is given by $F(x, y) = C$.

**Theorem**

*If $M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on a region $R$ in $xy-$plane, then the differential equation (5) is exact if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{on } R.$$

# Example



Prove that the following differential equations are exact and find their solutions

$$(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)dy = 0.$$

**Solution**

Here

$$\frac{\partial M}{\partial y} = -2xy - 2 = \frac{\partial N}{\partial x},$$

so $Eq$ (61) is exact. Then there exists a function $F$ of $x$ and $y$ such that

$$\frac{\partial F}{\partial x} = 2x^3 - xy^2 - 2y + 3.$$

and

$$\frac{\partial F}{\partial y} = -(x^2y + 2x).$$

From $Eq$ (62) we have

$$F(x,y) = \int (2x^3 - xy^2 - 2y + 3)dx = \frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2yx + 3x + g(y),$$

where $g$ will be determined from $Eq$ (62). The latter yields

$$\begin{aligned} -x^2y - 2x + g'(y) &= -x^2y - 2x, \\ g'(y) &= 0. \end{aligned}$$

Therefore $g(y) = C$, then the solution of the differential equation (61) is defined implicitly by

$$\frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2yx + 3x + C = 0.$$

# Example

$$\left[\cos x \ \ln(2y-8)+\frac{1}{x}\right]dx+\frac{\sin x}{y-4}dy=0, \ \ x\neq 0, \ \text{and} \ y>4.$$

**Solution.**

Here
$$\frac{\partial M}{\partial y}=\cos x\frac{2}{2y-8}=\cos x.\frac{1}{y-4}=\frac{\partial N}{\partial x}.$$

Thus, *Eq* (64) is exact, then there exists a function $F$ of $x$ and $y$ such that

$$\frac{\partial F}{\partial x} = M = \cos x \, \ln(2y - 8) + \frac{1}{x}.$$

$$\frac{\partial F}{\partial y} = N = \frac{\sin x}{y-4}.$$

From $Eq$ (66) we have $F(x,y) = \int \frac{\sin x}{y-4} dy = \sin x \; \ln(y-4) + g(x)$, where the function $g$ will be determined by $Eq$ (65)

$$\begin{aligned} \frac{\partial F}{\partial x} &= \cos x \; \ln(y-4) + g'(x) \\ &= \cos x \; \ln(2y-8) + \frac{1}{x} \\ &= \cos x \; \ln 2 + \cos x \; \ln(y-4) + \frac{1}{x}, \end{aligned}$$

hence

$$g'(x) = \frac{1}{x} + \cos x \ \ln 2 \ \textit{or} \ \ g(x) = \ln|x| + \sin x \ \ln 2 + C,$$

so the solution of the differential equation (62) is defined implicitly by

$$F(x, y) = \sin x \ \ln(y - 4) + \ln|x| + \sin x \ \ln 2 + C = 0,$$

$$F(x, y) = \sin x \ \ln(2y - 8) + \ln|x| + C = 0.$$

# Example

$$(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0, \quad y \neq 0.$$

**Solution.**

We have

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Then *Eq* (68) is exact and there exists a function $F$ of $x$ and $y$ such that

$$\frac{\partial F}{\partial x} = M = e^{2y} - y\cos xy.$$

$$\frac{\partial F}{\partial y} = N = 2xe^{2y} - x\cos xy + 2y.$$

Now from *Eq* (68) we deduce that

$$F(x, y) = xe^{2y} - \sin xy + g(y),$$

where the function $g$ will be determined from *Eq* (69)

$$\frac{\partial F}{\partial y} = 2xe^{2y} - x\cos xy + g'(y) = 2xe^{2y} - x\cos xy + 2y,$$

hence

$$g'(y) = 2y \text{ or } g(y) = y^2 + C,$$

So the solution of the differential equation (68) is defined implicitly by

$$F(x, y) = xe^{2y} - \sin xy + y^2 + C = 0.$$

# Example



Solve the initial value problem ($IVP$)

$$\begin{cases} \frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y \neq 0 \ \text{ and } x \neq \pm 1, \\ y(0) = 2. \end{cases}$$

**Solution.** The differential equation (71) can be written in the form

$$y(1 - x^2)dy + (-xy^2 + \cos x \sin x)dx = 0.$$

We have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -2xy,$$

then *Eq* (71) is exact and there exists a function $F$ of $x$ and $y$ such that

$$\frac{\partial F}{\partial x} = M = -xy^2 + \cos x \sin x.$$

$$\frac{\partial F}{\partial y} = N = y(1 - x^2).$$

Now from *Eq* (72) we have

$$F(x, y) = -\frac{1}{2}x^2y^2 + \frac{1}{2}\sin^2(x) + g(y),$$

where $g$ will be determined from *Eq* (73)

$$\frac{\partial F}{\partial y} = -x^2y + g'(y) = y - yx^2,$$

hence

$$g'(y) = y \text{ or } g(y) = \frac{1}{2}y^2 + C,$$

So the solution of the differential equation in (71) is defined implicitly by

$$F(x,y) = -\frac{1}{2}x^2y^2 + \frac{1}{2}\sin^2(x) + \frac{1}{2}y^2 + C = 0.$$

Now from the initial condition $y(0) = 2$, we deduce that $C = -2$, hence the solution of the $(IVP)$ is given by the curve

$$-\frac{1}{2}x^2y^2 + \frac{1}{2}\sin^2(x) + \frac{1}{2}y^2 - 2 = 0.$$

# Exercises



Test each of the following equations for exactness and solve it. If some of the equations are not exact, then use the appropriate method to solve them.

1. $(6x + y^2)dx + y(2x - 3y)dy = 0.$
2. $(2xy - 3x^2)dx + (x^2 + y)dy = 0.$
3. $(y^2 - 2xy + 6x)dx - (x^2 - 2xy + 2)dy = 0.$
4. $(x - 2y)dx + 2(y - x)dy = 0.$
5. $(2xy + y)dx + (x^2 - x)dy = 0.$

6. $(1+y^2)dx + (x^2y + y)dy = 0.$
7. $(1+y^2+xy^2)dx + (x^2y + y + 2xy)dy = 0.$
8. $(2xy - \tan y)dx + (x^2 - x\sec^2 y)dy = 0.$
9. $x(3xy - 4y^3 + 6)dx + (x^3 - 6x^2y^2 - 1)dy = 0.$

10. $(xy^2 + y - x)dx + x(xy + 1)dy = 0.$

Solve the following initial value problems

11. $$\begin{cases} (x-y)dx + (-x+y+2)dy = 0, \\ y(1) = 1. \end{cases}$$

# Integrating Factors

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0,$$

where $M$, $N$, $\frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous on a certain region $R$ in $xy$-plane. Suppose that $Eq$ (6) is not exact, that is

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ on } R.$$

## Definition

A function $h$ of $x$ and $y$ is called an integrating factor of *Eq* (6) if the differential equation

$$(h\ M\ )dx\ + (h\ N)\ dy = 0,$$

is exact, that is

$$\frac{\partial(hM)}{\partial y} = \frac{\partial(hN)}{\partial x} \text{ on } R,$$

where $h(x,y) \neq 0$ for all $(x,y) \in R$.

Since (2) is exact, we can solve it, and its solutions will also satisfy the differential equation (6).

As $h = h(x, y)$ is an integrating factor of *Eq* (6), then $h$ satisfies the partial differential equation

$$N\ h_x - M\ h_y\ = (M_y\ \ - N_x)\ h.$$

In general, it is very difficult to solve the partial differential *Eq* (81) without some restrictions on the functions $M$ and $N$ of the *Eq* (6). Suppose $h$ is a function of one variable, for example, say that $h$ depends only on $x$. In this case, $h_x = \frac{dh}{dx}$ and $h_y = 0$, so *Eq* (81) can be written as

$$\frac{dh}{dx} = \frac{M_y\ \ - N_x}{N}\ h.$$

We are still at an awkward situation if the quotient $\frac{M_y - N_x}{N}$ depends on both $x$ and $y$. However, if after all obvious algebraic simplifications are made, the quotient $\frac{M_y - N_x}{N}$ turns out depend solely on the variable $x$, then *Eq* (81) is a first -order ordinary differential equation. We can finally determine $h$ because *Eq* (81) is separable as well as linear. Then we have

$$h(x) = e^{\int (\frac{M_y - N_x}{N}) dx}.$$

In like manner, it follows from *Eq* (81) that if $h$ depends only the variable $y$, then

$$\frac{dh}{dy} = \frac{N_x - M_y}{M}\ h.$$

In this case, if $(N_x - M_y)\diagup M$ is a function of $y$ only, then we can solve *Eq* (83) for $h$.
We summarize the results for the differential equation

$$M(x,y)dx + N(x,y)dy = 0.$$

i) If $\dfrac{M_y - N_x}{N}$ is a function of $x$ only, then an integrating factor for $Eq$ (83) is

$$h(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$

ii) If $\dfrac{N_x - M_y}{M}$ is a function of $y$ only, then an integrating factor for $Eq$ (83) is

$$h(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

# Example



Find the solution of the differential equation

$$xydx + (2x^2 + 3y^2 - 20)dy = 0,$$

where $x \neq 0$ and $y > 0$.
**Solution.** We have

$$M = xy \text{ and } N = 2x^2 + 3y^2 - 20,$$

then $M_y = x$ and $N_x = 4x$, so $Eq$ (85) is not exact.

But

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20},$$

so this quotient depends on $x$ and $y$.But

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y} = g(y),$$

Then the integrating factor for *Eq* (85) is

$$h(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int g(y) dy} = e^{\int \frac{3}{y} dy} = e^{\ln y^3} = y^3.$$

Then we multiply the equation *Eq* (85) by

$$h\,(y) = y^3,$$

and we obtain

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3) dy = 0.$$

This equation is exact, because

$$M_y = N_x = 4xy^3.$$

So there exists a function $F$ of $x$ and $y$ satisfies

$$\begin{aligned}\frac{\partial F}{\partial x} &= M = xy^4.\\ \frac{\partial F}{\partial y} &= N = 2x^2y^3 + 3y^5 - 20y^3.\end{aligned}$$

Hence

$$F(x,y) = \int (xy^4)dx \implies F(x,y) = \frac{1}{2}x^2y^4 + g(y).$$

But

$$\frac{\partial F}{\partial y} = 2x^2y^3 + g'(y) = 2x^2y^3 + 3y^5 - 20y^3 \implies g'(y) = 3y^5 - 20y^3,$$

or

$$g(y) = \frac{1}{2}y^6 - 5y^4 + C.$$

Then the solution of the differential equation (85) is given by

$$F(x, y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + C = 0.$$

# Example

Solve the differential equation :

$$(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0, \;\; x(x + 2y) \neq 0.$$

**Solution.** Here

$$M = 4xy + 3y^2 - x, \;\; N = x^2 + 2xy,$$

so

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x + 6y \; - (2x + 2y) = 2(x + 2y).$$

Hence

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x).$$

Then the integrating factor for *Eq* (90) is

$$h(x) = e^{\int f(x)dx} = e^{2\ln|x|} = x^2.$$

Returning to the original *Eq* (90), we insert the integrating factor and obtain

$$(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy = 0,$$

where we know that *Eq* (91) must be an exact equation. Let us find the function $F$ of $x$ and $y$ by another method. We can put *Eq* (91) in the form

$$(4x^3y\ dx + x^4dy) + (3x^2y^2dx + 2x^3ydy) - x^3dx = 0,$$

hence

$$d\ (x^4y) + d\ (x^3y^2) + d\ (\frac{-1}{4}x^4) = d\ (x^4y + x^3y^2\frac{-1}{4}x^4) = 0,$$

so

$$d(F(x,y)) = d\ (x^4y + x^3y^2 - \frac{1}{4}x^4) = 0 \implies F(x,y) = \ x^4y + x^3y^2$$

is the solution of the differential equation (90).

# Example

Solve the differential equation

$$y(x+y+1)dx + x(x+3y+2)dy = 0, \quad y(x+y+1 \neq 0.$$

**Solution.** Here

$$M = yx + y^2 + y, N = x^2 + 3xy + 2x,$$

then

$$\begin{aligned}
\frac{\partial M}{\partial y} &= x+2y+1, \quad \frac{\partial N}{\partial x} = 2x+3y+2, \\
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} &= -x-y-1 = -(x+y+1), \\
\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) &= \frac{(x+y+1)}{y(x+y+1} = \frac{1}{y} = g(y),
\end{aligned}$$

so the integrating factor for *Eq* (93) is

$$h(y) = e^{\int g(y)dy} = e^{\int \frac{dy}{y}} = |y|\,.$$

It follows that if $y > 0$, then $h(y) = y$ and if $y < 0$, we have $h(y) = -y$. In other case *Eq* (93) becomes

$$(xy^2 + y^3 + y^2)dx + (x^2y + 3xy^2 + 2xy)dy = 0,$$

or

$$(xy^2dx + x^2ydy) + (y^3dx + 3xy^2dy) + (y^2dx + 2xydy) = 0,$$

$$d\ \left(\frac{1}{2}x^2y^2\right) + d\ (xy^3) + d\ (xy^2) = 0,$$

$$d\ (\ F(x,y) = d\ \left(\frac{1}{2}x^2y^2 + xy^3 + xy^2\right) = 0,$$

Then the solution of the differential equation (93) is

$$F(x,y) = \frac{1}{2}x^2y^2 + xy^3 + xy^2 + C = 0.$$

# Example



Find $k,\ n \in \mathbb{Z}$ such that $h(x, y) = x^k y^n$, is an integrating factor of the differential equation

$$y(x^3 - y)dx + -x(x^3 + y)dy = 0, \;\; x > 0, \;\; y > 0.$$

**Solution.**

$$(x^3y - y^2)dx - (x^4 + xy)dy = 0,$$

We have to find $k$ and $n$ such that the equation

$$(x^{k+3}y^{n+1} - y^{n+2}x^k)dx - (x^{k+4}y^n + x^{k+1}y^{n+1})dy = 0,$$

is exact, which means that

$$\begin{aligned} \frac{\partial M}{\partial y} &= (n+1)y^n x^{k+3} - (n+2)y^{n+1}x^k \\ &= \frac{\partial N}{\partial x} = -(k+4)x^{k+3}y^n - (k+1)x^k y^{n+1}, \end{aligned}$$

hence

$$(n+k+5)y^n x^{k+3} + (k-n-1)x^k y^{n+1} = 0,$$

which implies that

$$\left\{\begin{array}{l} n+k+5=0 \\ k-n-1=0 \end{array}\right| \Longrightarrow n=-3, \quad \text{and } k=-2.$$

So the differential equation

$$\left(\frac{x}{y^2} - \frac{1}{yx^2}\right)dx + \left(-\frac{x^2}{y^3} - \frac{1}{xy^2}\right)dy = 0,$$

is exact, and it is easy to see that the solution of $Eq$ (98) is given by

$$F(x,y) = \frac{x^2}{2y^2} + \frac{1}{xy} + C = 0.$$

# Exercises

Solve each of the following equations.

1. $(x^2 + y^2 + 1)dx + x(x - 2y)dy = 0.$
2. $y(2x - y + 1)dx + x(3x - 4y + 3)dy = 0.$
3. $(xy + 1)dx + x(x + 4y - 2)dy = 0.$
4. $(2y^2 + 3xy - 2y + 6x)dx + x(x + 2y - 1)dy = 0.$
5. $y^2dx + (3xy + y^2 - 1)dy = 0.$
6. $2(2y^2 + 5xy - 2y + 4)dx + x(2x + 2y - 1)dy = 0.$
7. $y(2x^2 - xy + 10dx + (x - y)dy = 0.$

In problems 8- 12, solve the given differential equation by finding an appropriate integrating factor.

8. $(2y^2 + 3x)dx + 2xydy = 0.$
9. $\cos x \; dx + (1 + \frac{2}{y}) \sin x \; dy = 0.$
10. $(10 - 6y + e^{-3x})dx - 2dy = 0.$
11. $(x^4 + y^4)dx - xy^3dy = 0.$
12. $(x^2 - y^2 + x)dx + 2xydy = 0.$

In problems 13 and 14, solve the given initial-value problem by finding an appropriate integrating factor.

13 $\begin{cases} xdx + (x^2y + 4y)dy = 0, \\ y(4) = 0. \end{cases}$

14 $\begin{cases} (x^2 + y^2 - 5)dx = (y + xy)dy, \\ y(0) = 1. \end{cases}$

Solve the exercise 15 by two methods.

15 $y(8x - 9y)dx + 2x(x - 3y)dy = 0.$

16 Find the value $k$ so that the given differential equation is exact.
$(y^3 + kxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0.$

# The General Solution of Linear Differential Equation



Consider the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Suppose that $P$ and $Q$ are continuous functions on an interval $a < x < b$ and $x = x_0$ is any number in that interval. If $y_0$ is an arbitrary real number, there exists a unique solution $y = y(x)$ of the differential equation (7) which satisfies the initial condition

$$y(x_0) = y_0.$$

Moreover, this solution satisfies *Eq* (7) throughout the entire interval $a < x < b$. It is easy to see that

$$h(x) = e^{\int P(x)dx}.$$

is an integrating factor for *Eq* (7) and the general solution of *Eq* (7) is given by

$$y \ h(x) = \int h(x) \ Q(x) \ dx \ + C.$$

Since $h(x) \neq 0$ for all $x \in (a, b)$ we can write

$$y(x) = e^{-\int P(x)dx} \left[ \int h(x) \ Q(x) \ dx \right] + Ce^{-\int P(x)dx}.$$

We can choose the constant $C$ so that $y = y_0$ when $x = x_0$.

# Example



Find the general solution of the differential equation

$$(1+x^2)\frac{dy}{dx}+xy+x^3+x=0.$$

**Solution.** *Eq* (104) can be written in the form $\frac{dy}{dx}+\frac{x}{1+x^2}y=-x$.. Then $h(x)=e^{\int \frac{x}{x^2+1}dx}=e^{\ln\sqrt{x^2+1}}=\sqrt{x^2+1}$, so

$$\begin{aligned} y\ h(x) &= y\sqrt{x^2+1}=\int h(x)\ Q(x)\ dx \\ &= -\int x\sqrt{x^2+1}dx=\frac{-1}{3}(1+x^2)^{\frac{3}{2}}+C. \end{aligned}$$

Hence the general solution of *Eq* (104) is

$$y(x) = -\frac{1}{3}(x^2+1) + \frac{C}{\sqrt{x^2+1}}.$$

The general solution of *Eq* (104) can be written as the sum of two solutions

$$y(x) = y_h + y_p,$$

where $y_h = \dfrac{C}{\sqrt{x^2+1}}$ is the general solution of $\dfrac{dy}{dx} + \dfrac{x}{1+x^2}y = 0$, and $y_p = -\dfrac{1}{3}(x^2+1)$ is a particular solution of the equation $\dfrac{dy}{dx} + \dfrac{x}{1+x^2}y = -x$.

# Example



Find the general solution of the differential equation

$$2(2xy + 4y - 3)dx + (x+2)^2 dy = 0, \quad x \neq -2.$$

**Solution.**
*Eq* (106) can be written in the form
$\dfrac{dy}{dx}(x+2)^2 + 4y(x+2) = 6$, or $\dfrac{dy}{dx} + \dfrac{4}{x+2}y = \dfrac{6}{(x+2)^2}$.

Then $h(x) = e^{\int \frac{4}{x+2} dx} = e^{4 \ln|x+2|} = (x+2)^4$, thus

$$y \; h(x) = y \; (x+2)^4 = \int h(x) Q(x) dx = \int 6(x+2)^2 dx = 2(x+2)$$

Hence the general solution of *Eq* (106) is

$$y(x) = \frac{2}{x+2} + C \frac{1}{(x+2)^4}.$$

# Example

Find the initial value problem ($IVP$)

$$(y - x + xy\cot x)dx + xdy = 0, \quad 0 < x < \pi,$$
$$y(\tfrac{\pi}{2}) = 0.$$

**Solution.**
We have $x\dfrac{dy}{dx} + y(1 + x\cot x) = x$, or $\dfrac{dy}{dx} + (\dfrac{1}{x} + \cot x)y = 1$.
Then

$$h(x) = e^{\int(\frac{1}{x}+\cot x)dx} = e^{\ln x + \ln(\sin x)} = x\sin x.$$

So the general solution of *Eq* (108) is

$$h(x)y = x\sin x \; y(x) \;= \int x \sin x \; dx \;= \; -x\cos x + \sin x + C,$$

or

$$y(x) = -\cot x \; + \frac{1}{x} + C\frac{1}{x\sin x}.$$

Now we use the condition $y(\frac{\pi}{2}) = 0$, to find the constant $C$. In fact

$$y(\frac{\pi}{2}) = -(0) + \frac{2}{\pi} + C\frac{2}{\pi} = 0 \Longrightarrow C = -1.$$

then the solution of the $(IVP)$ (108) is

$$y(x) = -\cot x \; + \frac{1}{x} - \frac{1}{x\sin x}.$$

# Example



Find the initial value problem ($IVP$)

$$\begin{cases} (x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}, \quad x > -1, \\ y(0) = 1. \end{cases}$$

**Solution.**
We have $\dfrac{dy}{dx} + (1 + \dfrac{1}{x+1})y = \dfrac{2x}{x+1}e^{-x}$. Then $h(x) = e^{\int (1+\frac{1}{x+1})dx} = e^{x+\ln(x+1)} = (x+1)e^{x}$, and the general solution of $Eq$ (110) is

$$h(x)y = (x+1)e^{x}y = \int h(x)Q(x)dx = \int 2xdx = x^2 + C,$$

or $y(x) = \frac{x^2}{x+1}e^{-x} + C\frac{1}{x+1}e^{-x}$. From the condition $y(0) = 1$, we deduce that $y(0) = 0 + C = 1 \Longrightarrow C = 1$. Hence the solution of $(IVP)$ (110) is

$$y(x) = \frac{x^2}{x+1}e^{-x} + \frac{1}{x+1}e^{-x}.$$

# Exercises

In exercises 1 through 9, find the general solution.

1. $(x^5 + 3y)dx - xdy = 0.$
2. $(2xy + x^2 + x^4)dx - (1 + x^2)dy = 0.$
3. $((y - \cos^2(x))dx + \cos x dy = 0, \quad 0 < x < \frac{\pi}{2}.$
4. $x^2 y' + xy = x + 1.$
5. $x\frac{dy}{dx} - y = x^2 \sin x.$

6. $x^2y' + x(x+2)y = e^x$.
7. $(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$.
8. $\frac{dy}{dx} - \frac{3}{x-1}y = (x-1)^4$.
9. $y' - \frac{x}{1+x^2} = -\frac{x}{1+x^2}y$.

In exercises 10 through 14, solve the initial value problem.

10 $$\begin{cases} y' - xy = (1 - x^2)e^{\frac{1}{2}x^2}, \\ y(0) = 0. \end{cases}$$

11 $$\begin{cases} (1 - x)\frac{dy}{dx} + xy = x(x - 1)^2, \\ y(5) = 24. \end{cases}$$

12 $$\begin{cases} (2x+3)y' = y + (2x+3)^{\frac{1}{2}}, \\ y(-1) = 0. \end{cases}$$

13 $$\begin{cases} (3xy+3y-4)dx + (x+1)^2 dy = 0, \\ y(0) = 1. \end{cases}$$

14 $\begin{cases} x(x^2+1)y' + 2y = (x^2+1)^3, \\ y(1) = -1. \end{cases}$

15 Solve the differential equation $(x+a)y' = bx - ny$, where $a$, $b$, and $n$ are constants with $n \neq 0$, $n \neq -1$.

16 Solve the equation of exercise (110) for the exceptional cases $n = 0$ and $n = -1$.

17 In the standard form

$$dy + Pydx = Qdx.$$

put $y = vw$, thus

$$w(dv + Pvdx) + vdw = Qdx.$$

then, by first choosing $v$ so that

$$dv + Pvdx = 0,$$

and later determining $w$, show how to complete the solution

$$dy + Pydx = Qdx.$$

# Bernoulli's Equation



Bernoulli's equation is a well known differential equation which has the general form

$$y' + P(x)y = Q(x)y^n,$$

where $n \in \mathbb{R}$.

1. If $n = 0$ then $Eq$ (8) is a linear first differential equation and we have discussed before.
2. If $n = 1$, $Eq$ (8) becomes a differential equation with separable variables, so we solve it.
3. Now we suppose that $n \neq 0$ and $n \neq 1$, we suppose also $y \neq 0$ on some interval $I = (a, b)$, then $Eq$ (8) can be written in the form

$$y^{-n}y' + P(x)y^{-n+1} = Q(x).$$

Now we put $u = y^{-n+1}$, then we have

$$u' = (-n+1)y^{-n}y',$$

so $Eq$ (3) becomes $\frac{1}{-n+1}u' + P(x)u = Q(x)$, , or

$$u' + (-n+1)P(x)u = Q(x)(-n+1),$$

is linear, and can be solved.

# Example



Solve the differential equation

$$y(6y^2 - x - 1)dx + 2xdy = 0, \quad x > 0.$$

**Solution.**
First we write *Eq* (121) in the form

$$y' - \frac{x+1}{2x}y = \frac{-3}{x}y^3,$$

so the obtained equation is a Bernoulli equation, where $n = 3$. Now suppose that $y \neq 0$ on some interval $I = (a, b)$, then *Eq* (121) can be written in the form

$$y'y^{-3} - \frac{x+1}{2x}y^{-2} = \frac{-3}{x},$$

and put

$$u = y^{-2} \implies u' = -2y^{-3}y',$$

hence *Eq* (122) becomes

$$u' + \frac{x+1}{x}u = \frac{6}{x}.$$

This equation is linear and the integrating factor for *Eq* (122) is

$$h(x) = e^{\int (1+\frac{1}{x})dx} = xe^x.$$

Then the solution of *Eq* (122) is

$$xe^x u = 6e^x + C,$$

so the solution of *Eq* (121) is

$$y^2(6 + Ce^{-x}) = x.$$

# Example



Write the differential equation

$$3(1+x^2)\frac{dy}{dx} = 2xy(y^3-1).$$

in the form of Bernoulli's equation an solve it, where $y \neq 0$ on some interval $I = (a, b)$.

**Solution.**

*Eq* (124) can be written in the form

$$y' + \frac{2x}{3(x^2+1)}y = \frac{2x}{3(x^2+1)}y^4.$$

So we have Bernoulli's equation with $n = 4$. We divide *Eq* (124) by $y^4$ and we get

$$y'y^{-4} + \frac{2x}{3(x^2+1)}y^{-3} = \frac{2x}{3(x^2+1)}.$$

Now we put $u = y^{-3}$, then

$$u' = -3y^{-4}y',$$

and *Eq* (125) becomes

$$u' - \frac{2x}{(x^2+1)}u = -\frac{2x}{(x^2+1)}.$$

*Eq* (125) is linear which has an integrating factor

$$h(x) = \frac{1}{x^2+1} \Longrightarrow \frac{1}{x^2+1}u = \frac{1}{x^2+1} + C.$$

Then the solution of *Eq* (124) is

$$y^3\left[1+(x^2+1)C\right] = 1.$$

# Example

Find the solution of the initial value problem

$$\begin{cases} (2y^3 - x^3)dx + 2xy^2 dy = 0, \quad x > 0, \\ y(1) = 1. \end{cases}$$

**Solution.**
The differential equation in the $(IVP)$ (127) can be written in the form

$$y' + \frac{1}{x}y = \frac{x^2}{2}y^{-2}.$$

So $Eq$ (127) is a Bernoulli equation with $n = -2$, and suppose that $y \neq 0$ on some interval $I = (a, b)$. From $Eq$ (127) we deduce that

$$y^2 y' + \frac{1}{x}y^3 = \frac{x^2}{2}.$$

Put

$$u = y^3 \implies u' = 3y^2y',$$

hence we have

$$\frac{1}{3}u' + \frac{1}{x}u = \frac{x^2}{2}.$$

or

$$u' + \frac{3}{x}u = \frac{3}{2}x^2.$$

*Eq* (128) is linear which has an integrating factor $h(x) = x^3$, then the solution of *Eq* (128) is

$$ux^3 = \frac{1}{4}x^6 + C.$$

so the solution of the differential equation is

$$y^3 = \frac{1}{4}x^3 + \frac{1}{x^3}C.$$

Now we use the condition $y(1) = 1$, then $C = \frac{3}{4}$, so the solution of the $(IVP)$ (127) is

$$y^3 = \frac{1}{4}x^3 + \frac{3}{4x^3}.$$