Statistical Methods – STAT 105

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Chapter 1
Discrete Random Variables and Distributions

Definition (Random Variable)

A random variable is a function that associates a real number with each element in the sample space.

Example

Toss two coins and let X denote the number of heads observed. Then

$$X \in \{0, 1, 2\}.$$

For example, the outcomes HT and TH both give $X = 1$.

Note: We use a capital letter, such as X, to denote a random variable and the corresponding lowercase letter, x, for one of its values.

Discrete and Continuous Random Variables

Discrete Random Variable

A random variable is called a discrete random variable if its set of possible outcomes is countable. In most practical problems, discrete random variables represent count data.

Examples

Number of defective items in a sample; number of customers arriving in one hour; number of heads in three coin tosses.

Continuous Random Variable

When a random variable can take on values on a continuous scale, it is called a continuous random variable. In most practical problems, continuous random variables represent measured data.

Examples

Height, weight, temperature, and waiting time.

Definition (Probability function)

The set of ordered pairs $(x, f(x))$ is a probability function, probability mass function, or probability distribution of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_{x} f(x) = 1$,
3. $P(X = x) = f(x)$.

Definition (cumulative distribution function)

The cumulative distribution function $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for} \quad -\infty < x < +\infty.$$

Definition (Mean of a Random Variable)

Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is

$$\mu = E(X) = \sum_x x\, f(x).$$

Theorem

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x).$$

Example

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$f(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

1. Find $E(X)$.
2. Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution

Simple calculations yield:

x	4	5	6	7	8	9
$f(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$.
$xf(x)$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{4}{3}$	$\frac{3}{2}$
$g(x)$	7	9	11	13	15	17
$g(x)f(x)$	$\frac{7}{12}$	$\frac{9}{12}$	$\frac{11}{4}$	$\frac{13}{4}$	$\frac{15}{6}$	$\frac{17}{6}$

1. $E(X) = \frac{1}{3} + \frac{5}{12} + \frac{3}{2} + \frac{7}{4} + \frac{4}{3} + \frac{3}{2} = 6.83$
2. The attendant's expected earnings for this particular time period is equal to:

$$E[g(X)] = \frac{7}{12} + \frac{9}{12} + \frac{11}{4} + \frac{13}{4} + \frac{15}{6} + \frac{17}{6} = 12.67.$$

Properties of the mean:

Theorem

Let X be a random variable. If a and b are constants, then $E(aX + b) = aE(X) + b$.

Theorem

The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

Example

Let X be a random variable with probability distribution as follows:

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Find the expected value of $Y = (X-1)^2$.

Solution We can solve it using two methods:

1) Simple calculations yield:

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$
$g(x)$	1	0	1	4
$f(x)g(x)$	$\frac{1}{3}$	0	0	$\frac{2}{3}$

Therefore, the expected value of Y is equal to:

$$E(Y) = E[g(X)] = 1.$$

2) Another solution: by using the properties of the mean theorems, $E(Y) = E((X-1)^2) = E(X^2) - 2E(X) + 1.(Hint!)$

$$E(g(x)) = \sum_{all\ x} g(x).f(x)$$

Theorems (Variance of Random Variable)

Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$Var(X) = \sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x).$$

or it can be written as:

$$\sigma_x^2 = \sigma^2 = E(X^2) - E(X)^2$$

$$Var(g(X)) = \sigma^2_{g(X)} = E[(g(x)-\mu_{g(x)})^2] = \sum_x (g(x)-\mu_{g(x)})^2 f(x).$$

The positive square root of the variance, σ, is called the standard deviation of X.

σ_x

Properties of the Variance:

$\sigma^2 \geq 0$

$Var(aX+b) = Var(aX) + Var(b)$
$= a^2 Var(X) + 0$

Theorem

Let X be a random variable. If a and b are constants, then $Var(aX+b) = a^2 Var(X)$.

Corollary

Setting $a = 1$, then $Var(X+b) = Var(X)$.

Corollary

Setting $b = 0$, then $Var(aX) = a^2 Var(X)$.

Example

Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution:

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

$$\sigma^2_{g(x)} = \sum_{\forall x} [g(x) - \mu_{g(x)}]^2 \cdot f(x)$$

Solution

1 Simple calculations yield

$$\sum_{\forall x} [g(x) - 6]^2 f(x)$$

$g(x) = 2x + 3$

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$
$g(x)$	3	5	7	9
$g(x)f(x)$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{7}{2}$	$\frac{9}{8}$

Therefore, The expected value of $g(X)$ is equal to

$$E(g(x)) = \sum_{\forall x} g(x) \cdot f(x)$$

$$E[g(X)] = \frac{3}{4} + \frac{5}{8} + \frac{7}{2} + \frac{9}{8} = 6$$

So, the variance of $g(X) = 2X + 3$ is equal to

$$\sigma^2_{g(x)} = (3-6)^2 * \frac{1}{4} + (5-6)^2 * \frac{1}{8} + (7-6)^2 * \frac{1}{2} + (9-6)^2 * \frac{1}{8} = 4,$$

and the standard deviation of $g(X)$ is equal to: $\sigma_{g(x)} = \sqrt{4} = 2.$

2 Another solution: By using the properties of the variance

$$Var(2X+3) = 2^2 Var(X) = 4Var(X) = 4[E(X^2) - E(X)^2]$$

$E(X^2) = \sum_{\forall x} x^2 f(x)$

$E(x) = \sum_{\forall x} x f(x)$

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$
x^2	0	1	4	9

Therefore, $E[X^2] = \frac{1}{8} + 2 + \frac{9}{8} = \frac{26}{8}$, and $E[X] = \frac{1}{8} + 1 + \frac{3}{8} = \frac{12}{8}$.
Then, the variance of $g(X) = 2X + 3$ is equal to

$$\sigma^2_{g(X)} = 4\left[\frac{26}{8} - (\frac{12}{8})^2\right] = 4.$$

Definition (Discrete Uniform Random Variable)

A random variable X has a discrete uniform distribution if it can take one of k values, say $x_1, x_2, \ldots, x_k$, and all values are equally likely. Its probability mass function is:

$$f(x) = P(X = x) = \begin{cases} \frac{1}{k}, & x = x_1, x_2, \ldots, x_k \\ 0, & \text{otherwise.} \end{cases}$$

Note: Each possible value has probability $1/k$.

Theorem

The expected value (mean) and variance of the discrete uniform distribution are:

$$\mu = E(X) = \sum_{i=1}^{k} \frac{x_i}{k}, \text{ and } \sigma^2 = \frac{1}{k} \sum_{i=1}^{k} [x_i - E(X)]^2.$$

Example

Suppose that you select a ball from a box containing 6 balls labeled $1, 2, \cdots, 6$. Let $X=$ the number that is observed when selecting a ball. Find $E(X)$ and $Var(X)$.

Solution

The probability distribution of X is:

$$P(X = x) = \begin{cases} \frac{1}{6}, & x = 1, 2, \ldots, 6 \\ 0, & \text{otherwise.} \end{cases}$$

The expected value:

$$\mu = E(X) = \sum_{i=1}^{k} \frac{x_i}{k} = \frac{1+2+3+4+5+6}{6} = 3.5.$$

The variance:

$$\sigma^2 = \frac{1}{k} \sum_{i=1}^{k} [x_i - E(X)]^2 = \frac{(1-3.5)^2 + \cdots + (6-3.5)^2}{6} = 2.92.$$

Definition (Bernoulli Process)

The Bernoulli process possesses the following properties:

1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

Definition (Binomial Distribution)

Let X be the number of successes in n independent Bernoulli trials, each having the same probability of success p. Then X has a binomial distribution. Its probability mass function is:

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2 \ldots, n.$$

We denote the binomial distribution with parameters n and p by $X \sim Bin(n, p)$ and $\binom{n}{x} = \frac{n!}{x!(n-x)!}$. $= {}^nC_x \rightarrow$ in calculator

The Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of binomial distribution $Bin(n, p)$ is:

$$F_X(x) = P(X \leq x) = \sum_{i=0}^{x} \binom{n}{i} p^i q^{n-i}, \quad x = 0, 1, \ldots, n.$$

Theorem

The mean and variance of the binomial distribution $Bin(n, p)$ are

$$\mu = n\,p \text{ and } \sigma^2 = n\,p\,q.$$

Example

If the mean and the variance of a binomial distribution are 10 and 5 respectively, then:

$E(X) = np = 10$

$Var(X) = npq = 5$

$10q = 5$

$\Rightarrow q = \frac{1}{2}$; $p = \frac{1}{2}$

$n\frac{1}{2} = 10 \rightarrow n = 20$

1. Determine the probability mass function.
2. Calculate the probability $P(X = 0)$.
3. Calculate the probability $P(X \geq 2)$.

Solution

1. By solving $E(X) = np = 10$ and $Var(X) = np(1 - p) = 5$, we get $p = 0.5$ and $n = 20$. The probability mass function is:

$$P(X = x) = \binom{20}{x} \left(0.5\right)^x \left(0.5\right)^{20-x}, \; x = 0, 1, \cdots, 20$$

calculator

Σ

2. $P(X = 0) = \binom{20}{0} \left(0.5\right)^0 \left(0.5\right)^{20} = 0.5^{20}$.
3. $P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)] = 1 - 0.00002 = 0.99998$.

Example

Suppose that the probability that a person dies when he or she contracts a certain disease is 0.4. A sample of 10 persons who contracted this disease is randomly chosen. Find the following:

1. The probability that exactly 4 persons will die.
2. The probability that less than 3 persons will die. <3
3. The probability that more than 8 persons will die. $>$
4. The expected number of persons who will die.
5. The variance of the number of persons who will die.

$n = 10$

$p = 0.4$

$q = 0.6$

Solution X: number of person dies when he or she contracts a certain diseas.

The probability mass function is:

$$P(X = x) = \binom{10}{x} (0.4)^x (0.6)^{10-x}, \; x = 0, 1, \cdots, 10$$

1. $P(X = 4) = \binom{10}{4} (0.4)^4 (0.6)^{10-4} = 0.251$

2.

$$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$
$$= \sum_{x=0}^{2} \binom{10}{x} (0.4)^x (0.6)^{10-x} = 0.167.$$

3.

$$P(X > 8) = P(X = 9) + P(X = 10)$$
$$= \sum_{x=9}^{10} \binom{10}{x} (0.4)^x (0.6)^{10-x} = 0.0017.$$

4. $E(X) = np = (10)(0.4) = 4$
5. $Var(X) = np(1 - p) = (10)(0.4)(0.6) = 2.4$

Definition

Suppose a finite population of size N contains K items classified as successes and $N - K$ items classified as failures. If a sample of size n is selected without replacement and X is the number of successes in the sample, then X has a hypergeometric distribution.

There are two methods of selection:

1. Selection with replacement: If we select the elements of the sample at random and with replacement, then $X \sim Bin(n, p)$; where $p = \frac{K}{N}$
2. Selection without replacement: When selection is made without replacement, X has a hypergeometric distribution with parameters N, n, and K. We write $X \sim \text{Hypergeometric}(N, K, n)$.

The probability mass function for hypergeometric random variable X is:

$$f(x) = P(X = x) = \begin{cases} \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} & \max\{0, n-(N-K)\} \le x \le \min\{K, n \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

The mean and variance of the hypergeometric distribution $h(x; N, n, K)$ are

p p q

$$\mu = n\frac{K}{N} \text{ and } \sigma^2 = n\frac{K}{N}\left(1 - \frac{K}{N}\right)\frac{N-n}{N-1}.$$

N K $N-K$

Example

Suppose there are 50 officers, 10 female officers and 40 male officers. Suppose 20 of them will be selected for promotion. Let X represent the number of female promotions. Find:

$n = 20$

1. The probability that exactly one female is found in the sample.
2. The expected value (mean) and the variance of the number of females in the sample.

N n K

$h(50, 20, 10)$

Solution

- Note that the binomial distribution doesn't apply here, as the officers are without replacement once they are drawn. In other words, the trials are not independent events.
- X has a hypergeometric distribution with $N = 50$, $n = 20$, and $K = 10$; i.e. $X \sim h(x; N, n, K) = h(x; 50, 20, 10)$.

$$P(X = x) = \begin{cases} \dfrac{\binom{10}{x}\binom{50-10}{20-x}}{\binom{50}{20}} & x = 0, 1, 2, \cdots, 10 \\ 0, & \text{otherwise.} \end{cases}$$

1. The probability that exactly one female is found in the sample is:

$$f(1) = P(X = 1) = \frac{\binom{10}{1}\binom{40}{19}}{\binom{50}{20}} = 0.0279$$

2. The expected value (mean) is $E(X) = n\frac{K}{N} = 20 \times \frac{10}{50} = 4$.
3. The variance is
$\sigma^2 = n\,\frac{K}{N}\left(1 - \frac{K}{N}\right)\frac{N-n}{N-1} = 20 \times \frac{10}{50} \times (1 - \frac{10}{50}) \times \frac{50-20}{50-1} = 1.9592$

(Binomial Approximation Theorem)

If the sample size n is small relative to the population size N, then a binomial distribution $Bin(n, p = \frac{K}{N})$ can be used to approximate the hypergeometric distribution $h(x; N, n, K)$.

$N=5000$ $K=1000$ $n=10$ $p=\frac{1000}{5000}$

Example

A manufacturer of automobile tires reports that among a shipment of 5000 sent to a local distributor, 1000 are slightly blemished. If one purchases 10 of these tires at random from the distributor, what is the probability that exactly 3 are blemished?

$p=\frac{1}{5}=0.2$

Solution

Since the population size $N = 5000$ is large relative to the sample size $n = 10$, we shall approximate the desired probability by using the binomial distribution. The probability of obtaining a blemished tire is 0.2. Therefore, the probability of obtaining exactly 3 blemished tires is

$$h(3; \overset{N}{5000}, \overset{n}{10}, \overset{K}{1000}) \approx Bin(\overset{n}{10}, p = \overset{0.2}{\frac{1000}{5000}}) = \binom{10}{3} (0.2)^3 (0.8)^7 = 0.2013.$$

Definition

Let X be the number of occurrences of an event in a fixed interval of time, area, volume, or other unit. X is a Poisson random variable with parameter $\lambda > 0$ if its probability mass function is

$$P(x, \lambda) = P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

where e is an irrational number approximately equal to 2.71828 and λ is the average number of occurrences per interval unit.

Theorem $X \sim \text{Poisson}(\lambda)$

If a random variable X has a Poisson distribution, then both the mean and the variance of X are λ.

$$\mu = \lambda \; \textit{and} \; \sigma^2 = \lambda$$

Example

The mean number of accidents per month at a certain intersection is 3. $\lambda = 3$ for 1 month $\Rightarrow X \sim Poisson(3) \Rightarrow f(x) = \frac{e^{-3} 3^x}{x!}$ $x = 0, 1, 2, \ldots$

1. What is the probability that in any given month 4 accidents will occur at this intersection?
2. What is the probability that more than 4 accidents will occur in any given month at the intersection?
3. What is the probability that 4 accidents will occur in 5 months?

$= 1 - \sum_{x=0}^{4} \frac{e^{-3} 3^x}{x!} =$

Solution

1. $f(4) = P(X = 4) = e^{-3}\frac{3^4}{4!} = 0.168.$
2. $P(X > 4) = 1 - P(X \leq 4) = 1 - [P(X = 0) + \cdots + P(X = 4)] = 1 - [\sum_{x=0}^{4} e^{-3}\frac{3^x}{x!}] = 0.1847.$
3. Since the average number of accidents at a certain intersection per month is 3, thus the average number of accidents in 5 months is 15. Let X represent the number of accidents in 5 months, $f(x) = P(X = x) = e^{-15}\frac{15^x}{x!}, \quad x = 0, 1, 2, \ldots.$ Then, $f(4) = P(X = 4) = e^{-15}\frac{15^4}{4!} = 0.00065.$

To Find new λ^*

$\lambda = 3 \longrightarrow 1$ month

$\lambda^* = ? \longrightarrow 5$ month

$\lambda^* = 3 \times 5 = 15$

Theorem (Approximation)

Let X be a binomial random variable with probability distribution $B(n, p)$. When n is large ($n \to +\infty$), and p small ($p \to 0$), then the Poisson distribution can be used to approximate the binomial distribution $B(n, p)$ by taking $\lambda = np$.

$n \longrightarrow \infty$

$p \longrightarrow 0$

Binomial $\approx$ Poisson

Example

In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

1. What is the probability that in any given period of 400 days there will be an accident on one day?
2. What is the probability that there are at most three days with an accident?

Solution $X \sim Binomial(400, 0.005) \Rightarrow X \sim Poisson(2 \text{ for } 1 \text{ day})$

Let X be a binomial random variable with $n = 400$ and $p = 0.005$. Thus, $np = 2$. Use the Poisson approximation,

$$f(x) = \frac{e^{-2} 2^x}{x!}$$

$x = 0, 1, 2, \ldots$

1. $$P(X = 1) = e^{-2}\frac{2^1}{1!} = 0.271.$$

2. $$P(X \leq 3) = \sum_{x=0}^{3} e^{-2}\frac{2^x}{x!} = 0.857.$$