Elementary Methods-First-Order Differential Equations

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Definition

1. An ordinary differential equation (ODE) is an equation that contains one or more derivatives of the unknown function. A function that satisfies the equation is called a solution of the differential equation. The most general form of an ordinary differential equation is:

$$F(x, y(x), y^{(1)}(x), \dots, y^{(n)}(x)) = 0, \tag{1}$$

where $F\colon \Omega \longrightarrow \mathbb{R}$ is a funcfion defined on an an open set Ω of $\mathbb{R}^{n+2}$.

2. The highest derivative that appear in an ordinary differential equation is called the order of that ordinary differential equation. Under a suitable restriction on the function F, the equation (1) can be solved explicitly for $y^{(n)}$ in terms of the other $n+1$ variables $x, y, y', y'', \dots, y^{(n-1)}$, to obtain

Definition

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}). \tag{2}$$

We shall assume that this is always possible. For example, the equation

$$x(y')^2 + 4y' - 6x^2 = 0, \quad x \neq 0.$$

has two different equations,

$$y' = \frac{-2 + \sqrt{4 + 6x^3}}{x}, \quad \text{or} \quad y' = \frac{-2 - \sqrt{4 + 6x^3}}{x}.$$

Definition

3. If the ordinary differential equation (1) is in the form

$$a_0(x)y^{(n)}(x) + \ldots + a_{n-1}(x)y^{(1)}(x) + a_n(x)y(x) = f(x) \quad (3)$$

   is is called linear. If such representation is not possible, then we say that the given ordinary differential equation is nonlinear.
4. If $f = 0$ in (3), the equation is called homogeneous linear ordinary differential equation.

Definition

5. If the functions $a_0, a_1, \ldots, a_n$ are constants, the equation (3) is called linear ordinary differential equation with constant coefficients.
6. Similarly, if the functions $a_0, a_1, \ldots, a_n$ are constants and $f = 0$, the equation (3) is called linear homogeneous ordinary differential equation with constant coefficients.

Examples

1. The equation $\frac{dy}{dx} = e^x + x^2 + 1$ is linear with constant coefficients.
2. The equation $(x^2 + y^2)dx - 4xy\ dy = 0$ is nonlinear.
3. $\frac{\partial u}{\partial t} = 4(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$ is not an ordinary differential equation.
4. The equation $L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} = E\omega\cos(\omega t)$ is linear with constant coefficients.
5. The equation $\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^3 + y^2 = 0$, is of order two. It also referred to as second order equation.

Definition

1. A function y defined on an interval I is called a solution of the differential equation (2) provided that the n derivatives of the function exist on the interval I and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for every x in I. In this case the solution $y(x)$ is called an explicit solution of the differential equation (2).

2. A relation $F(x, y) = 0$ is said to be an **implicit solution** of the ordinary differential equation (2) on an interval I, if the relation defines implicitly a function $y = \phi(x)$ which satisfies the differential equation (2) on an interval I.

Example

1. Prove that $y = e^{2x}$ is a solution of the equation
$$y'' + y' - 6y = 0, \forall\ x \in \mathbb{R}.$$
2. Verify that $y = x^3 e^x$ is a solution of the differential equation
$$xy'' - 2(x+1)y' + (x+2)y = 0, \quad x > 0.$$
3. Verify that the relation
$$F(x, y) = x^2 + y^2 - 25 = 0.$$
defines an implicit solution of the differential equation
$$\frac{dy}{dx} = -\frac{x}{y},\ y \neq 0.$$
on the interval $(-5, 5)$.

Example

Eliminate the arbitrary constants c_1 and c_2 from the relation

$$y = c_1 e^{-2x} + c_2 e^{3x}.$$

to obtain a differential equation.

Solution.

$$y' = -2c_1 e^{-2x} + 3c_2 e^{3x},$$

$$y'' = 4c_1 e^{-2x} + 9c_2 e^{3x}.$$

We get $y'' + 2y' = 15c_2 e^{3x}$ and $y' + 2y = 5c_2 e^{3x}$. Hence $y'' + 2y' = 3(y' + 2y)$, or $y'' - y' - 6y = 0$.

Example

Eliminate the arbitrary constant a from the equation

$$(x-a)^2+y^2=a^2, x\neq 0$$

Solution.
Direct differentiation of the relation yields

$$2(x-a)+2yy'=0,$$

from which

$$a=x+yy',$$

therefore, using the original equation, we find that

$$(yy')^2+y^2=(x+yy')^2,$$

or

$$2xyy'+x^2-y^2=0,$$

The equation $(x-a)^2+y^2=a^2$, may be put in the form

$$x^2+y^2-2ax=0.$$

For $x\neq 0$, we get $\dfrac{x^2+y^2}{x}=2a$. The differentiation of both sides leads to

$$x(2xdx+2ydy)-(x^2+y^2)dx=0,$$

or

$$(x^2-y^2)dx+2xydy=0.$$

Example

Eliminate B and α from the relation

$$x(t) = B\cos(\omega t + \alpha).$$

where B and α are two arbitrary constants and ω is a fixed constant.
We first obtain two derivatives of x with respect to t

$$\frac{dx}{dt} = -\omega B\sin(\omega t + \alpha),$$

$$\frac{d^2x}{dt^2} = -\omega^2 B\ \cos(\omega t + \alpha).$$

We get

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

Example

Eliminate the arbitrary constant c from the family of curves

$$cxy + c^2x + 4 = 0.$$

At once we get

$$c(y + xy') + c^2 = 0.$$

Since $c \neq 0$, we have $c = -(y + xy')$, and substitution into the original equation leads us to the result

$$x^3(y')^2 + x^2yy' + 4 = 0.$$

An equation involving a parameter, as well as one both of the coordinates of a plane, may represent a family of curves corresponding to each value of the parameter. For instance, the equation

$$(x-c)^2+(y-c)^2=2c^2,$$

or

$$x^2+y^2-2c(x+y)=0,$$

may be interpreted as the equation of a family of circles, each having its center on the line $y=x$ and each passing through the origin.

The result of the elimination of these constants is a differential equation which represent the family.
We get $x^2 + y^2 = 2c(x + y)$. We differentiate these equation and we get $2(x + yy')(x + y) = (1 + y')(x^2 + y^2)$ and

$$(x^2 + 2xy - y^2) + y'(2xy - x^2 - y^2) = 0.$$

Example

Find a differential equation satisfied by the family of parabolas having their vertices at the origin and their foci on the $y-$axis. An equation of this family of parabolas is $y = ax^2$, where $a \neq 0$ is an arbitrary constant, then $y' = 2ax$. It follows that $xy' - 2y = 0$, is the differential equation of the family.

Example

Find the differential equation of the family of circles having their centers on the y-axis.
These circles have the equation in the form

$$x^2 + (y - b)^2 = c^2.$$

We shall eliminate both b and c and obtain a second-order differential equation for this family of circles. $x + (y - b)y' = 0$, and $1 + y''(y - b) + (y')^2 = 0$. Then $y' + y''y'(y - b) + (y')^3 = 0$. By substitution, we get

$$y' - xy'' + (y')^3 = 0.$$

Finally we give some mathematical model of differential equations which arise in physical sciences.

1. **Free falling stone**

$$\frac{d^2s}{dt^2} = -g, \tag{4}$$

where s is a distance or height and g is the gravity acceleration.

2. **Spring vertical displacement**

$$m\frac{d^2y}{dt^2} = -ky, \tag{5}$$

where y is the displacement, m is the mass and k is the spring constant.

3. **Motion of simple Pendulum**

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \quad (6)$$

where θ is an angular displacement, L is the length of the rod, and g is the acceleration due to the gravity.

4. **RLC-circuit,Kirchoff's Second Law**

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E, \qquad (7)$$

where q is the charge on capacitor, L is inductance, c is the capacitance, R is the resistance, and E is the voltage.

5. **Deformation of a beam**

$$EI\frac{d^4y}{dx^4} = f(x), \tag{8}$$

where E is the Young's modulus, I is the moment of Inertia and $f(x)$ is the load per unit length.

6. **Growth and Decay**

$$\frac{dP}{dt} = KP, \quad P(t_0) = P, \tag{9}$$

where P is a given quantity and K is the constant of proportionality.

7. **Newton's of Heating and Cooling**

$$\frac{dT}{dt} = k(T - T_s), \tag{10}$$

where $\frac{dT}{dt}$ is the rate of the body, $T - T_s$ is temperature difference between the body T and its surrounding T_s and k is the constant of proportionality.
The first derivative can be interpreted for example as the speed of motion, or the change of temperature, the rate of change in the population,...etc. Simple physical examples are models (9) and (10) above. Also the second derivative may represent an acceleration, for example in Newton's law of motion $F = ma$, where F is the algebraic sum of acting forces and a, the acceleration of the system motion, that is the rate of change of the speed, hence represented by a second derivative of motion. As models see (5), (6) and (7) above.