

# Chapter 5

## One and Two-Sample Tests of Hypotheses

Department of Statistics and Operations Research



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  - Tests of two-sided hypotheses on the mean
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ثَبَتَ      نَفَى      تَهَفَّنَى

Consider a population with some unknown parameter  $\Theta$ . We are interested in testing (confirming or denying) some conjectures about  $\Theta$ . For example, we might be interested in testing the conjecture that

$$\Theta = \Theta_0 \text{ where } \Theta_0 \text{ is a given value.}$$

## Definition

A **statistical hypothesis** is an assertion or conjecture concerning one or more populations.

تأكيد

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We usually test the null hypothesis:

الفرضية الصفرية  $H_0 : \Theta = \Theta_0$  (Null Hypothesis).

against one of the following alternative hypotheses:

H<sub>0</sub> or H<sub>1</sub>  $H_1 : \begin{cases} \Theta \neq \Theta_0, \\ \Theta > \Theta_0, \\ \Theta < \Theta_0 \end{cases}$  (Alternative Hypothesis or Research Hypothesis)

*دو*

Possible situations in testing a statistical hypothesis:

	$H_0$ is true	$H_0$ is false
Accepting $H_0$	Correct Decision	Type II error ( $\beta$ )
Rejecting $H_0$	Type I error ( $\alpha$ )	Correct Decision

## Definition

Rejection of the null hypothesis when it is true is called a **type I error**.

$H_0$

The probability of committing a type I error, also called the level of significance, is denoted by  $\alpha$ .

$$\Pr(\text{type I error}) = \Pr(\text{Rejecting } H_0 | H_0 \text{ is true}) = \alpha$$

## Definition

Nonrejection of the null hypothesis when it is false is called a **type II error**.

$H_0$

The probability of committing a type II error, denoted by  $\beta$ , is impossible to compute unless we have a specific alternative hypothesis.

$$\Pr(\text{type II error}) = \Pr(\text{Accepting } H_0 | H_0 \text{ is false}) = \beta$$

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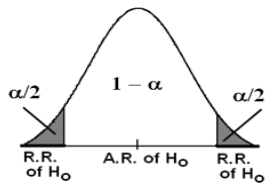


# Tests of <sup>two</sup> one-sided hypotheses on the mean $Z : \neq$

Let  $X_1, X_2, \dots, X_n$  representing a random sample from a distribution with mean  $\mu$  and variance  $\sigma^2 > 0$ . Consider first the hypothesis

$$H_0 : \mu = \mu_0,$$

$$H_1 : \mu \neq \mu_0.$$



Two-sided alternative

## Test statistic (T.S.)

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

and

$$Z_{1-\frac{\alpha}{2}}$$

$$\Pr\left(-z_{\alpha/2} \leq \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) = 1 - \alpha$$

من جدول في التوزيع

$z_{1-\alpha/2}$

### Theorem Test Procedure for a Single Mean (Variance Known)

If  $-z_{\alpha/2} < z < z_{\alpha/2}$ , do not reject  $H_0$ . Rejection of  $H_0$ , of course, implies acceptance of the alternative hypothesis  $\mu \neq \mu_0$ . With this definition of the critical region, it should be clear that there will be probability  $\alpha$  of rejecting  $H_0$  (falling into the critical region) when, indeed,  $\mu = \mu_0$ .

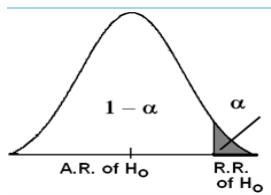
في الـ  $z_{\alpha/2}$

# Tests of one-sided hypotheses on the mean

$Z : ">"$

$$H_0 : \mu = \mu_0,$$

$$H_1 : \mu > \mu_0.$$



one-sided alternative

## Test statistic (T.S.)

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

and

$$\Pr\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq z_{1-\alpha}\right) = 1 - \alpha$$

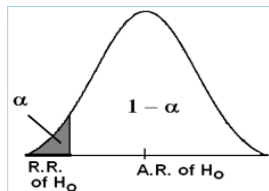
*(Note: A handwritten green arrow points from  $z_{1-\alpha}$  to the critical value  $z_{1-\alpha}$  in the equation.)*

في التوزيع

# Tests of one-sided hypotheses on the mean $Z : "<"$

$$H_0 : \mu = \mu_0,$$

$$H_1 : \mu < \mu_0.$$



one-sided alternative

## Test statistic (T.S.)

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

and

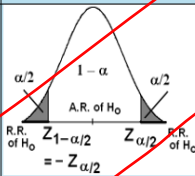
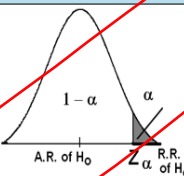
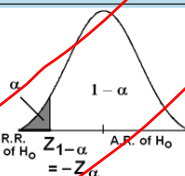
$$\Pr\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq -z_{\alpha}\right) = 1 - \alpha$$

*Handwritten note:  $z_{1-\alpha}$  (with an arrow pointing to  $-z_{\alpha}$ )*

*Handwritten note:  $z_{1-\alpha}$  في الـ  $z_{\alpha}$*

# Conclusion

## Test Procedure

<b>Hypotheses</b>	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$
<b>Test Statistic (T.S.)</b>	$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim N(0,1)$		
<b>R.R. and A.R. of <math>H_0</math></b>			
<b>Decision</b>	Reject $H_0$ (and accept $H_1$ ) at the significance level $\alpha$ if:		
	$Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Two-Sided Test	$Z > Z_{\alpha}$ One-Sided Test	$Z < -Z_{\alpha}$ One-Sided Test

## Example

A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution  $\rightarrow$  الرجوع لطايف الحلول

1)

$$\begin{cases} H_0 : \mu = 70 \text{ years,} \\ H_1 : \mu > 70 \text{ years.} \end{cases}$$

2)  $\alpha = 0.05$ .

3) Test statistic:

$$z = (\bar{x} - \mu_0) / (\sigma / \sqrt{n}) = (71.8 - 70) / (8.9 / \sqrt{100}) = 2.02.$$

4) Critical region:  $z > z_{\alpha}$ , where  $z_{\alpha} = z_{0.05} = 1.645$ .

5) Decision: since  $z = 2.02 > 1.645$ , reject  $H_0$  and conclude that the mean life span today is greater than 70 years.

The P-value corresponding to  $z = 2.02$  is given by the area on the right under the density of standard normal distribution. Using Table A.3, we have  $P\text{-value} = P(Z > 2.02) = 0.0217$ . As a result, the evidence in favor of  $H_1$  is even stronger than that suggested by a 0.05 level of significance.

### Example

A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that  $\mu = 8$  kilograms against the alternative that  $\mu \neq 8$  kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.01 level of significance.

Solution →

الدرجة المطلوبة

1)

$$\begin{cases} H_0 : \mu = 8 \text{ kilograms,} \\ H_1 : \mu \neq 8 \text{ kilograms.} \end{cases}$$

2)  $\alpha = 0.01$ .

3) Critical region:  $z > z_{\alpha/2}$  and  $z < -z_{\alpha/2}$ , where  $z = (\bar{x} - \mu_0)/(\sigma/\sqrt{n})$  and  $z_{\alpha/2} = 2.575$ .

4) Computations:  $\bar{x} = 7.8$ ,  $\sigma = 0.5$ ,  $n = 50$ , hence  $z = (\bar{x} - \mu_0)/(\sigma/\sqrt{n}) = (7.8 - 8)/(0.5/\sqrt{50}) = -2.83$

5) Decision: since  $z = -2.83 < -2.575$ , hence reject  $H_0$  and conclude that the average breaking strength is not equal to 8 but is, in fact, less than 8 kilograms.



Since the test in this example is two tailed, the desired P-value is twice the area of the left of  $z = -2.83$ . Therefore, using standard normal table, we have

$$\text{P-value} = 2P(Z > |2.83|) = 2P(Z > 2.83) = 0.0046 < 0.01$$

which allows us to reject the null hypothesis that  $\mu = 8$  kilograms at a level of significance smaller than 0.01.

# Tests on a Single Sample (Variance Unknown) on one mean T :

**Result:**(Single Mean (Variance Unknown) For the two-sided hypothesis

$$\begin{cases} H_0 : \mu = \mu_0, \\ H_1 : \mu \neq \mu_0. \end{cases}$$

We reject  $H_0$  at significance level  $\alpha$  when the computed t-statistic

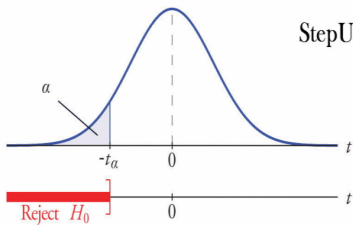
$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

exceeds  $t_{\frac{\alpha}{2}, n-1}$  or is less than  $-t_{\frac{\alpha}{2}, n-1}$ .

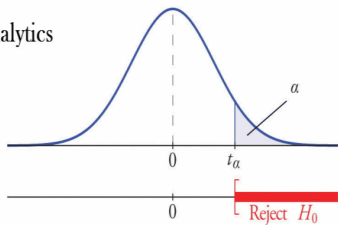
For  $H_1 : \mu > \mu_0$ , rejection results when  $t > t_{\alpha, n-1}$ . For  $H_1 : \mu < \mu_0$ , the critical region is given by  $t < -t_{\alpha, n-1}$ .

ملاحظة: في التوزيع

$$H_a : \mu < \mu_0$$

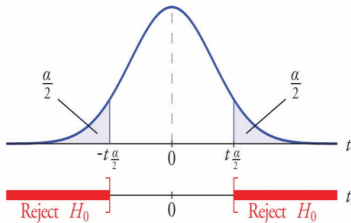


$$H_a : \mu > \mu_0$$



### StepUp Analytics

$$H_a : \mu \neq \mu_0$$



موجود فی التالیف

## Example

The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of 46 kilowatt hours per year. If a random sample of 12 homes indicates that vacuum cleaners use an average of 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours, does this suggest at the 0.05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually? Assume the population of kilowatt hours to be normal.

Solution

الرجوع لطرف التعليل

1)

$$\begin{cases} H_0 : \mu = 46 \text{ kilowatt hours,} \\ H_1 : \mu < 46 \text{ kilowatt hours.} \end{cases}$$

2)  $\alpha = 0.05$ .

3) Critical region:  $t < -t_{\alpha, n-1}$ ,  
where  $T = (\bar{X} - \mu_0)/(S/\sqrt{n})$   
with 11 degrees of freedom and  
 $t_{\alpha, n-1} = 1.796$ .

4) Computations:  $\bar{X} = 42$  kilowatt hours,  $s = 11.9$  kilowatt hours,  
and  $n = 12$ .

Hence,  $t = (42 - 46)/(11.9/\sqrt{12}) = -1.16$ .

5) Since  $t > -1.796$ , we do not reject  $H_0$  and conclude that the average number of kilowatt hours used annually by home vacuum cleaners is not significantly less than 46.

Also  $\dagger$  P-value =  $\Pr(T < -1.16) = \Pr(T > 1.16) \approx 0.135$ .

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The two-sided hypothesis on two means can be written generally as  $H_0 : \mu_1 = \mu_2$ . For  $\sigma_1$  and  $\sigma_2$  known, the test statistic is given by

$$Z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

موجود في التوزيع

That is, reject  $H_0$  in favor of  $H_1 : \mu_1 \neq \mu_2$  if  $z > z_{\alpha/2}$ , and  $z < -z_{\alpha/2}$ .

$z_{1-\frac{\alpha}{2}}$

One-tailed critical regions are used in the case of the one-sided alternatives. The reader should, as before, study the test statistic and be satisfied that for, say,  $H_1 : \mu_1 > \mu_2$ , the signal favoring  $H_1$  comes from large values of  $z$ . Thus, the upper-tailed critical region applies.

## Unknown But Equal Variances *For testing on two means T:*

If we assume that both distributions are normal and that  $\sigma_1 = \sigma_2 = \sigma$ , the two-sample t-test may be used. The test statistic is given by the following test procedure.

**Result:** (Two-Sample Pooled t-Test)

For the two-sided hypothesis

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2,$$

*Handwritten note in red:*  $\sigma_1 = \sigma_2 = \sigma$

we reject  $H_0$  at significance level  $\alpha$  when the computed t-statistic

$$T = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{1/n_1 + 1/n_2}}$$

where

$$S_p^2 = \frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

exceeds  $t_{\alpha/2, n_1+n_2-2}$  or is less than  $-t_{\alpha/2, n_1+n_2-2}$ .



ملاحظة في الدفتر

For  $H_1 : \mu_1 > \mu_2$ , reject  $H_0 : \mu_1 = \mu_2$  when  $t > t_{\alpha, n_1+n_2-2}$ . For  $H_1 : \mu_1 < \mu_2$ , reject  $H_0 : \mu_1 = \mu_2$  when  $t < -t_{\alpha, n_1+n_2-2}$ .

### Example

An experiment was performed to compare the abrasive wear of two materials. Twelve pieces of material 1 were tested and ten pieces of material 2 were similarly tested. The samples of material 1 gave an average (coded) wear of 85 units with a sample standard deviation of 4, while the samples of material 2 gave an average of 83 with a sample standard deviation of 5. Can we conclude at the 0.05 level of significance that the abrasive wear of material 1 exceeds that of material 2? Assume the populations to be approximately normal with equal variances.

Solution  $\rightarrow$

البيانات طلب المعلومات

Let  $\mu_1$  and  $\mu_2$  represent the population means of the abrasive wear for material 1 and material 2, respectively. 1)

$$\begin{cases} H_0 : \mu_1 = \mu_2, \\ H_1 : \mu_1 > \mu_2. \end{cases}$$

2)  $\alpha = 0.05$ .

3) Critical region:  $t > t_{\alpha, n_1+n_2-2}$ , where

$$T = \frac{\bar{X}_1 - \bar{X}_2}{S_p \sqrt{1/n_1 + 1/n_2}}$$

with  $\nu = 20$  degrees of freedom and  $t_{\alpha, n_1+n_2-2} = t_{0.05, 20} = 1.725$ .

4) Computations:

$$\begin{array}{l} \bar{X}_1 = 58 \quad S_1 = 4 \quad n_1 = 12 \\ \bar{X}_2 = 81 \quad S_2 = 5 \quad n_2 = 10 \end{array}$$

Hence

$$S_p = \sqrt{\frac{(11)(16) + (9)(25)}{12 + 10 - 2}} = 4.478$$

$$t = \frac{(85 - 83)}{4.478\sqrt{1/12 + 1/10}} = 1.04 < 1.725$$

+

$$\text{P-value} = \Pr(T > 1.04) \approx 0.16. (\text{See Table A.4.})$$

5) Decision: Do not reject  $H_0$ . We are unable to conclude that the abrasive wear of material 1 exceeds that of material 2.

## Paired Observations for testing on two means $T$ :

Testing of two means can be accomplished when data are in the form of paired observations, as discussed in Chapter 4. The statistical test for two means  $\mu_1$  and  $\mu_2$  in the situation with paired observations is based on the random variable

$$T = \frac{\bar{D} - \mu_D}{S_D / \sqrt{n}}$$

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where  $\bar{D}$  and  $S_D$  are random variables representing the sample mean and standard deviation of the differences of the observations in the experimental units. As in the case of the pooled t-test, the assumption is that the observations from each population are normal. This two-sample problem is essentially reduced to a one-sample problem by using the computed differences  $d_1, d_2, \dots, d_n$ . Critical regions are constructed using the t-distribution with  $n - 1$  degrees of freedom.

## Example **Blood Sample Data**

A study was conducted to examine the influence of the drug succinylcholine on the circulation levels of androgens in the blood. Blood samples were taken from wild, free-ranging deer immediately after they had received an intramuscular injection of succinylcholine administered using darts and a capture gun. A second blood sample was obtained from each deer 30 minutes after the first sample, after which the deer was released. The levels of androgens at time of capture and 30 minutes later, measured in nanograms per milliliter (ng/mL), for 15 deer are given in Table 6.2. Assuming that the populations of androgen levels at time of injection and 30 minutes later are normally distributed, test at the 0.05 level of significance whether the androgen concentrations are altered after 30 minutes.

## Androgen (ng/mL)

Deer	At Time of Injection	30 Minutes after Injection	$d_i$
1	2.76	7.02	4.26
2	5.18	3.10	-2.08
3	2.68	5.44	2.76
4	3.05	3.99	0.94
5	4.10	5.21	1.11
6	7.05	10.26	3.21
7	6.60	13.91	7.31
8	4.79	18.53	13.74
9	7.39	7.91	0.52
10	7.30	4.85	-2.45
11	11.78	11.10	-0.68
12	3.90	3.74	-0.16
13	26.00	94.03	68.03
14	67.48	94.03	26.55
15	17.04	41.70	24.66

Solution  $\rightarrow$  الرجوع لطايف الحل

Let  $\mu_1$  and  $\mu_2$  be the average androgen concentration at the time of injection and 30 minutes later, respectively. We proceed as follows:

1.  $H_0 : \mu_1 = \mu_2$  or  $\mu_D = \mu_1 - \mu_2 = 0$ .
2.  $H_1 : \mu_1 \neq \mu_2$  or  $\mu_D = \mu_1 - \mu_2 \neq 0$ .
3.  $\alpha = 0.05$ .
4. Critical region:  $t < -t_{\alpha/2, n-1}$  and  $t > t_{\alpha/2, n-1}$ , where  $t = \frac{\bar{D}}{S_D/\sqrt{n}}$  with  $\nu = 14$  degrees of freedom and  $t_{\alpha/2, n-1} = 2.145$ .
5. Computations: The sample mean and standard deviation for the  $d_i$  are

$$\bar{D} = 9.848 \quad \text{and} \quad S_D = 18.474$$

Therefore

$$t = \frac{9.848}{18.474/\sqrt{15}} = 2.06$$

6. Hence  $-2.145 < t = 2.06 < 2.145$ . Though the t-statistic is not significant at the 0.05 level, from Table A.4,

$$^+ P = P(|T| > 2.06) \approx 0.06$$

As a result, there is no evidence that there is a difference in mean circulating levels of androgen.



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Tests of hypotheses concerning proportions are required in many areas. We now consider the problem of testing the hypothesis that the proportion of successes in a binomial experiment equals some specified value. That is, we are testing the null hypothesis  $H_0$  that  $p = p_0$ , where  $p$  is the parameter of the binomial distribution. The alternative hypothesis may be one of the usual one-sided or two-sided alternatives:

$$p < p_0 \quad p > p_0 \quad \text{or} \quad p \neq p_0$$

We know that if  $np_0 \geq 5$  and  $n(1 - p_0) \geq 5$ , then the random variable  $\hat{P}$  is approximately a normal distribution with mean  $p_0$  and standard deviation  $\sigma_{\hat{P}} = \sqrt{p_0(1 - p_0)/n}$ .

موجود في التوزيع

موجود فی الٹا فرم

The **z-value** for testing  $p = p_0$  is given by

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

Hence, for a two-tailed test at the  $\alpha$ -level of significance, the critical region is  $z < -z_{\alpha/2}$  or  $z > z_{\alpha/2}$ . For the one-sided alternative  $p < p_0$ , the critical region is  $z < -z_{\alpha}$ , and for the alternative  $p > p_0$ , the critical region is  $z > z_{\alpha}$ .

$z_{1-\alpha/2}$

$z_{1-\alpha}$

## Example

A commonly prescribed drug for relieving nervous tension is believed to be only 60% effective. Experimental results with a new drug administered to a random sample of 100 adults who were suffering from nervous tension show that 70 received relief. Is this sufficient evidence to conclude that the new drug is superior to the one commonly prescribed? Use a 0.05 level of significance.

Solution  $\rightarrow$

الدواء يتفوق الدواء القديم

1)

$$\begin{cases} H_0 : p = 0.6, \\ H_1 : p > 0.6. \end{cases}$$

2)  $\alpha = 0.05$ .

3) Critical region:  $Z > z_\alpha$ , where  $z_\alpha = 1.645$ . Then, the critical region:  $z > 1.645$ .

4) Computations:  $x = 70, n = 100, \hat{p} = 70/100 = 0.7$ , and

$$z = \frac{0.7 - 0.6}{\sqrt{\frac{(0.6)(0.4)}{100}}} = 2.04$$

$$z = 2.04 > 1.645$$

$$P\text{-value} = \Pr(Z > 2.04) < 0.0207.$$

5) Decision: Reject  $H_0$  and conclude that the new drug is superior.

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Situations often arise where we wish to test the hypothesis that two proportions are equal. That is, we are testing  $p_1 = p_2$  against one of the alternatives  $p_1 < p_2$ ,  $p_1 > p_2$ , or  $p_1 \neq p_2$ . The statistic on which we base our decision is the random variable  $\hat{P}_1 - \hat{P}_2$ . When  $H_0 : p_1 = p_2 (= p)$  is true, we know that

$$Z = \frac{\hat{P}_1 - \hat{P}_2}{\sqrt{pq(1/n_1 + 1/n_2)}}$$

To compute a value of  $Z$ , however, we must estimate the parameters  $p$  and  $q$  that appear in the radical. Under  $H_0$ , both  $\hat{P}_1$  and  $\hat{P}_2$  are estimators of  $p$ .

هو لور فبا  
الله خير

We use the pooled estimate of the proportion  $p$ , which is

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

المتوسط  
المزيج

where  $x_1$  and  $x_2$  are the numbers of successes in each of the two samples. Substituting  $\hat{p}$  for  $p$  and  $\hat{q} = 1 - \hat{p}$  for  $q$ , the  $z$ -value for testing  $p_1 = p_2$  is determined from the formula

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}(1/n_1 + 1/n_2)}}$$

The critical regions for the appropriate alternative hypotheses are set up as before, using critical points of the standard normal curve.



## Example

A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. To determine if there is a significant difference in the proportions of town voters and county voters favoring the proposal, a poll is taken. If 120 of 200 town voters favor the proposal and 240 of 500 county residents favor it, would you agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters? Use an  $\alpha = 0.05$  level of significance.

Solution →

المجموع طالب الكليات

Let  $p_1$  and  $p_2$  be the true proportions of voters in the town and county, respectively, favoring the proposal.

$\hat{p}_1 = x_1/n_1 = 120/200 = 0.6$ ,  $\hat{p}_2 = x_2/n_2 = 240/500 = 0.48$ , and the pooled estimate

$$\hat{p} = (x_1 + x_2)/(n_1 + n_2) = (120 + 240)/(200 + 500) = 0.51.$$

1)

$$\begin{cases} H_0 : p_1 = p_2 \\ H_1 : p_1 > p_2. \end{cases}$$

2)  $\alpha = 0.05$ .

3) The test statistic

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}(1/n_1 + 1/n_2)}} = \frac{0.60 - 0.48}{(0.51)(0.49)(1/200 + 1/500)} = 2.9$$

4) Critical region:  $z > 1.645$ . P-value =  $P(Z > 2.9) = 0.0019$ .

5) Decision: Reject  $H_0$  and agree that the proportion of town voters favouring the proposal is higher than the proportion of county voters.

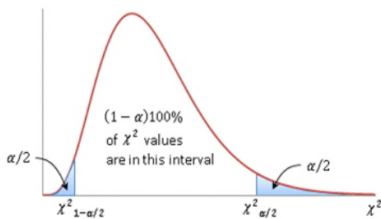
- 1 Introduction
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- 6 Two Samples: Tests on Two Proportions
- 7 One- and Two-Sample Tests Concerning Variances

In this section, we are concerned with testing hypotheses concerning population variances or standard deviations. Let us first consider the problem of testing the null hypothesis  $H_0$  that the population variance  $\sigma^2$  equals a specified value  $\sigma_0^2$  against one of the usual alternatives  $\sigma^2 < \sigma_0^2$ ,  $\sigma^2 > \sigma_0^2$ , or  $\sigma^2 \neq \sigma_0^2$ . If we assume that the distribution of the population being sampled is normal, the chi-squared value for testing  $\sigma^2 = \sigma_0^2$  is given by

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

موجه في الدرس

where  $n$  is the sample size,  $s^2$  is the sample variance, and  $\sigma_0^2$  is the value of  $\sigma^2$  given by the null hypothesis. If  $H_0$  is true,  $\chi^2$  is a value of the chi-squared distribution with  $\nu = n - 1$  degrees of freedom.



سؤال في التوزيع

Figure 5-2: Figure 5.2

Hence, for a two-tailed test at the  $\alpha$ -level of significance, the critical region is  $\chi^2 < \chi^2_{1-\alpha/2}$  or  $\chi^2 > \chi^2_{\alpha/2}$  (see figure 5.2). For the one-sided alternative  $\sigma^2 < \sigma_0^2$ , the critical region is  $\chi^2 < \chi^2_{1-\alpha}$ , and for the one-sided alternative  $\sigma^2 > \sigma_0^2$ , the critical region is  $\chi^2 > \chi^2_{\alpha}$ .

## Example

A manufacturer of car batteries claims that the life of the company's batteries is approximately normally distributed with a standard deviation equal to 0.9 year. If a random sample of 10 of these batteries has a standard deviation of 1.2 years, do you think that  $\sigma > 0.9$  year? Use a 0.05 level of significance.

Solution  $\rightarrow$

الواجب في طلب الحل

1)

$$\begin{cases} H_0 : \sigma^2 = 0.81, \\ H_1 : \sigma^2 > 0.81. \end{cases}$$

2)  $\alpha = 0.05$ .

3) Critical region: The null hypothesis is rejected when  $\chi^2 > 16.919$ , where  $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$ , with  $\nu = 9$  degrees of freedom.

4) Computations:  $s^2 = 1.44$ ,  $n = 10$ , and

$$\chi^2 = \frac{(9)(1.44)}{0.81} = 16.0, P \approx 0.07.$$

5) Decision: The  $\chi^2$ -statistic is not significant at the 0.05 level. However, based on the P-value 0.07, there is evidence that  $\sigma > 0.9$ .

Now let us consider the problem of testing the equality of the variances  $\sigma_1^2$  and  $\sigma_2^2$  of two populations. That is, we shall test the null hypothesis  $H_0$  that  $\sigma_1^2 = \sigma_2^2$  against one of the usual alternatives  $\sigma_1^2 < \sigma_2^2$ ,  $\sigma_1^2 > \sigma_2^2$ , or  $\sigma_1^2 \neq \sigma_2^2$ . For independent random samples of sizes  $n_1$  and  $n_2$ , respectively, from the two populations, the  $f$ -value for testing  $\sigma_1^2 = \sigma_2^2$  is the ratio

$$f = \frac{s_1^2}{s_2^2}$$

موجود في التوزيع

where  $s_1^2$  and  $s_2^2$  are the variances computed from the two samples. If the two populations are approximately normally distributed and the null hypothesis is true, then the ratio  $f = s_1^2/s_2^2$  is a value of the  $F$ -distribution with  $\nu_1 = n_1 - 1$  and  $\nu_2 = n_2 - 1$  degrees of freedom. Therefore, the critical regions of size  $\alpha$  corresponding to the one-sided alternatives  $\sigma_1^2 < \sigma_2^2$  and  $\sigma_1^2 > \sigma_2^2$  are, respectively,  $f < f_{1-\alpha}(\nu_1, \nu_2)$  and  $f > f_{\alpha}(\nu_1, \nu_2)$ . For the two-sided alternative  $\sigma_1^2 \neq \sigma_2^2$ , the critical region is  $f < f_{1-\alpha/2}(\nu_1, \nu_2)$  or  $f > f_{\alpha/2}(\nu_1, \nu_2)$ .

## Example

In testing for the difference in the abrasive wear of the two materials in Example 166, we assumed that the two unknown population variances were equal. Were we justified in making this assumption? Use a 0.10 level of significance.



Solution

الرجوع طلب الحل

Let  $\sigma_1^2$  and  $\sigma_2^2$  be the population variances for the abrasive wear of material 1 and material 2, respectively.

1)

$$\begin{cases} H_0 : \sigma_1^2 = \sigma_2^2, \\ H_1 : \sigma_1^2 \neq \sigma_2^2. \end{cases}$$

2)  $\alpha = 0.10$ .

3) Critical region: We have  $f_{0.05}(11, 9) = 3.11$ , and, by using Theorem 99, we find  $f_{0.95}(11, 9) = \frac{1}{f_{0.05}(9, 11)} = 0.34$ . Therefore, the null hypothesis is rejected when  $f < 0.34$  or  $f > 3.11$ , where  $f = s_1^2/s_2^2$  with  $\nu_1 = 11$  and  $\nu_2 = 9$  degrees of freedom.

4) Computations:  $s_1^2 = 16$ ,  $s_2^2 = 25$ , hence  $f = 16/25 = 0.64$ .

5) Decision: Do not reject  $H_0$ . Conclude that there is insufficient evidence that the variances differ.