CHAPTER 5

GRAPH COLORING

When we mention the word "graph" in this chapter, we mean a simple, finite and symmetric graph.

Graph Coloring

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We will consider the graphs are non null.

1 Definitions and Examples:

Definition 1.1

1.

   ⋆ *A* k**-vertex coloring** *(or, simply, a* k**-coloring***) of a graph* $G = (V, E)$ *is a mapping* $c : V \to S$*, where* S *is a set of* k *elements (these elements are called* **colors***).*
   ⋆ *Thus, a* k*- coloring is an assignment of* k *colors to the vertices of* G*.*
   ⋆ *Usually,* $S = \{1, 2, ..., k\}$*.*

2. *A* k*-coloring of a graph* G *is* **proper** *if no two adjacent vertices are assigned the same color.*

3.

   ⋆ *A* **stable set** *of a graph* $G = (V, E)$ *is a subset* X *of* V *such that, the induced subgraph* $G[X]$ *is empty.*

   ⋆ *The* **maximum size of the stable sets** *of a graph* G *is denoted: "*$\alpha(G)$*".*

4.

   ⋆ *A* **clique** *of a graph* $G = (V, E)$ *is a subset* X *of* V *such that, the induced subgraph* $G[X]$ *is complete.*

   ⋆ *The* **maximum size of the cliques** *of a graph* G *is denoted: "*$\omega(G)$*".*

5. *A graph is* k**-colorable** *if it has a proper* k*-coloring.*

6. *Given a graph G, the minimum k for which G is k-colorable is called the "***chromatic number of** G**"***; it is denoted:* $\chi(G)$.

Remarks 1.2

1.

$\star$ *Each graph $G = (V, E)$ admits a proper coloring; indeed, a proper n-coloring where $n = |V|$.*
$\star$ *A k-coloring of a graph $G = (V, E)$ is a partition $\{V_1, ..., V_k\}$ of V where V_i denotes the (possibly empty) set of vertices assigned color i. The V_i's are the color classes.*

$\star$ *A proper k-coloring of a graph $G = (V, E)$ is a k-coloring in which each color class is a* **stable set**.

2.

$\star$ *(A graph G is 1-colorable) if and only if (G is empty).*

$\star$ *(A graph G is 2-colorable) if and only if (G is bipartite).*
In particular case, *a tree is 2-colorable (tree is a bipartite graph).*

3.

*We are only concerned with proper colorings. So, we refer to a proper coloring as a "***coloring***" and to a proper k-coloring as a "k***-coloring***".*

4.

(a) *If H a subgraph (non null) of a graph G, then $\chi(H) \leq \chi(G)$.*
(b) *For each graph G, $\chi(G) \geq \omega(G)$.*
(c) *Let $X_1, ..., X_k$ be the connected components of a disconnected graph G, and let $G_i = G[X_i]$; for $i \in \{1, ..., k\}$. Clearly, $\chi(G) = max\{\chi(G_i); 1 \leq i \leq k\}$.*
(d) *Given a graph G, $\chi(G)$ is the smallest positive integer k for which there is a partition $\{V_1, ..., V_k\}$ of V, on k nonempty stable sets of G.*

Example 1.3

1. **Example 1:**

(a) $\chi(K_p) = p; \ \forall p \geq 1$.
(b) $\chi(D_p) = 1; \ \forall p \geq 1$.
(c) $\chi(C_3) = 3$ *and* $\chi(C_4) = 2$.
In general:$\chi(C_{2p+1}) = 3; \ \forall p \geq 1$, *and* $\chi(C_{2p}) = 2; \ \forall p \geq 2$.
(d) $\chi(T) = 2$; *pour tout tree* T.

2. **Example 2:**

   Find the chromatic number of following graph
   $G = (\{a,b,c,d,e,f\}, \{\{a,b\},\{a,c\},\{a,e\},\{b,c\},\{b,d\},\{b,f\},\{c,d\},\{c,e\},\{d,f\},\{f,e\}\})$.

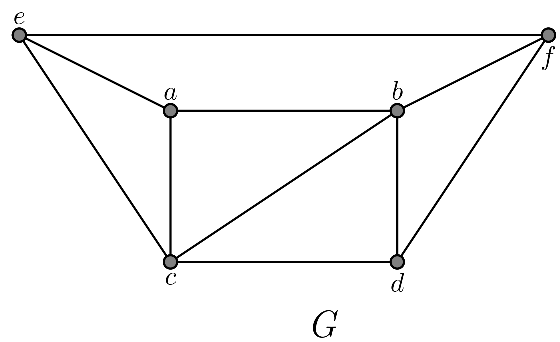
   

   Figure (a)

   ⋆ *As* $\{a,b,c\}$ *is a clique of* G*, then* $\chi(G) \geq 3$.
   ⋆ *Consider the subsets* V_1, V_2, *and* V_3, *of* $V(G) = \{a,b,c,d,e,f\}$*, defined by:* $V_1 = \{a,d\}$, $V_2 = \{b,e\}$, *and* $V_3 = \{c,f\}$.

   Clearly, $\{V_1, V_2, V_3\}$ *is a partition of* $V(G)$*, on non empty stable sets of* G*. So,* $\chi(G) \leq 3$.
   Thus, $\chi(G) = 3$*, and a* 3*-coloring of* G *is giving by: For* $i \in \{1,2,3\}$*, the vertices in* V_i *are assigned color* i.

3. **Application 1: (The Scheduling Problem)**

   The students at a certain university have annual examinations in all courses they take. Naturally,examinations in different courses cannot be held concurrently if the courses have students in common. How can all the examinations be organized in as few parallel sessions as possible?

   Solution
   Consider the graph G *whose vertices are the courses, two courses being joined by an edge if they give rise to a conflict.*
   Thus, stable sets of G *correspond to conflict-free groups of courses.*
   It ensues that the required minimum number of parallel sessions is $\chi(G)$.

4. **Application 2: (Chemical Storage)**

   A company manufactures n chemicals $C_1, C_2, ..., C_n$*. Certain pairs of these chemicals are incompatible and would course explosions if brought into contact with each other. As a precautionary measure, the company wishes to divide its warehouse into compartments, and store incompatible chemicals in different compartments.*
   What is the least number of compartments into which the warehouse should be partitioned?

   Solution
   Consider the graph G *whose vertices are* $C_1, C_2, ..., C_n$*, two chemicals being joined by an edge if they are incompatible.*
   Clearly, the least number of compartments into which the warehouse should be partitioned in equal to $\chi(G)$.

2 Some results

Theorem 2.1

Given a graph G we have: $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.

Proof.

$\rightarrow$ We know that: $\omega(G) \leq \chi(G)$
$\rightarrow$ To prove that: $\chi(G) \leq \Delta(G) + 1$, consider an integer $k > \Delta(G) + 1$ and let $\mathcal{C}_1$ be a k-coloring of G. The colors are: $1, 2, ..., k$.

N. B: It sufices to prove that there is a $(k-1)$-coloring $\mathcal{C}_2$ of G.

• Consider a vertex v of G assigned color k.
As, $|N_G(v)| \leq \Delta(G) < k - 1$, then there is a color $i \in \{1, 2, ..., k-1\}$ such that i is not used by the vertices in $N_G(v)$. Then, we may assign the color i for the vertex v, and we obtain a coloring of G. We may use this procedure to all vertices assigned color k by the k-coloring $\mathcal{C}_1$. Thus, we obtain a $(k-1)$-coloring $\mathcal{C}_2$ of G.

Remark 2.2

$\star$ *This result is better possible, because: For $G \in \{K_p, C_{2p+1}\}$, $\chi(G) = \Delta(G) + 1$ (where $p \geq 1$).*

$\star$ *But, we have the following result of Brook; which will state without proof:*

Theorem 2.3 *(***Brook's Theorem***)*

If G is a connected graph and it is neither an odd cycle nor a complete graph, then:
$\chi(G) \leq \Delta(G)$.

Remark 2.4 *(***"The four color problem"***)*

Consider a map divided into countries. We assign a color to each country, so that adjacent countries (i.e. countries sharing some common boundary) are assigned different colors.
What is the least number of colors required to color all countries in the map.

Problem

No map would require more than four colors (1852).

N. B *This problem was solved by Appel and Haken, in* (1976). *But the proof uses computer programs for many cases.*
There is no purely mathematical proof.!!!
This result is then:

Theorem: (The four color Theorem)

If G is a planar graph, then: $\chi(G) \leq 4$.

Here we prove the five color Theorem.

Theorem 2.5 The five color Theorem

If G is a planar graph, then: $\chi(G) \leq 5$.

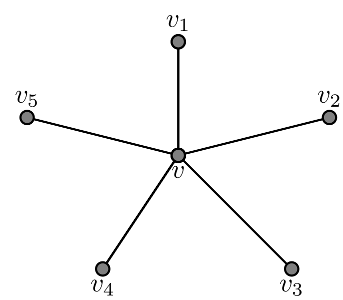
Proof.

By induction on $p = |V(G)| \geq 1$.

$\star$ For $p = 1$, the result is trivial.

$\star$ Let $k \geq 2$ be an integer and assume that all planar graph with $p = k-1$ vertices have chromatics number at most 5, and let G be a planar graph of order k. We know that $\delta(G) \leq 5$. Let v be a vertex with $d_G(v) = \delta(G)$. By induction hypothesis, the induced subgraph $G - v$ has a 5-coloring. Denoting the colors by: $1, 2, 3, 4,$ and 5. If one of these colors is not used in coloring the vertices adjacent with v, then we may assign that color to v, and we obtain a 5-coloring of G.

Thus, we may assume that: $d_G(v) = 5$ and that all the five colors are used by the vertices adjacent with v. Let us draw G as plane graph, and suppose v_1, v_2, v_3, v_4, v_5 are the 5 vertices adjacent with v, arranged cyclically about v, and suppose v_i has been colored by i; $(\forall i \in \{1, 2, 3, 4, 5\})$.



N. B: We will show that it is possible to recolor certain vertices of $G - v$, **including a vertex adjacent to** v, so a color becomes available for v.

$\rightarrow$ Consider the subgraph H of G defined by: $H = G[V_{1,3}]$ where $V_{1,3}$ denotes the set of vertices of $G - v$ which have been colored 1 or 3.

Case 1: If v_1 and v_3 are **different connected component** X_1 and X_3, respectively, of H.

We interchange the two colors of the vertices of X_1, and then we obtain another 5-coloring of $G-v$. However, this 5-coloring of $G - v$ assigns the color 3 to both v_1 and v_3. Thus, we may now assign the color 1 to the vertex v, producing a 5-coloring of G. So, $\chi(G) \leq 5$.

Case 2: If v_1 and v_3 are in the **same connected components** X of H.

In this case, there is a v_1v_3- path P in H. Thus, P is also a v_1v_3- path P of $G - v$, whose vertices are all colored 1 or 3. We add to P, the path (v_3, v, v_1) and we obtain a cycle in G which either encloses v_2 or encloses v_4 and v_5:

Thus, as G drawn as a plane graph, there is no v_2v_4-path in $G - v$ whose vertices are colored 2 or 4. So, in the subgraph $H' = G[V_{2,4}]$ where $V_{2,4}$ denotes the set of vertices of $G - v$ which have been colored 2 or 4, v_2 and v_4 are in the **different connected components** X_2 and X_4, respectively, of H'. As before, we interchange the two colors of the vertices in X_2, and we obtain a new 5-coloring of $G - v$ in which both v_2 and v_4 are colored 4. Hence, we may now assign the color 2 to the vertex v, and we obtain a 5-coloring of G. Thus, $\chi(G) \leq 5$.