# CHAPTER 5

# GRAPH COLORING

**When we mention the word "graph" in this chapter, we mean a simple, finite and symmetric graph.**

# Graph Coloring

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**We will consider the graphs are non null.**

## 1 Definitions and Examples:

**Definition 1.1**

1. ⋆ *A* $k$**-vertex coloring** *(or, simply, a* $k$**-coloring***) of a graph* $G = (V, E)$ *is a mapping* $c : V \to S$*, where* $S$ *is a set of* $k$ *elements (these elements are called* **colors***).*
   ⋆ *Thus, a* $k$*- coloring is an assignment of* $k$ *colors to the vertices of* $G$*.*
   ⋆ *Usually,* $S = \{1, 2, ..., k\}$*.*

2. *A* $k$*-coloring of a graph* $G$ *is* **proper** *if no two adjacent vertices are assigned the same color.*

3. ⋆ *A* **stable set** *(or* **independent set***) of a graph* $G = (V, E)$ *is a subset* $X$ *of* $V$ *such that, the induced subgraph* $G[X]$ *is empty.*

   ⋆ *The* **maximum size of the stable sets** *of a graph* $G$ *is denoted: "*$\alpha(G)$*".*

4. ⋆ *A* **clique** *of a graph* $G = (V, E)$ *is a subset* $X$ *of* $V$ *such that, the induced subgraph* $G[X]$ *is complete.*

   ⋆ *The* **maximum size of the cliques** *of a graph* $G$ *is denoted: "*$\omega(G)$*".*

5. *A graph is* $k$**-colorable** *if it has a proper* $k$*-coloring.*

6. *Given a graph $G$, the minimum $k$ for which $G$ is $k$-colorable is called the "***chromatic number of** $G$**"**; *it is denoted:* $\chi(G)$.

**Remarks 1.2**

1.

⋆ *Each graph* $G = (V, E)$ *admits a proper coloring; indeed, a proper* $n$*-coloring where* $n = |V|$.
⋆ *A* $k$*-coloring of a graph* $G = (V, E)$ *is a partition* $\{V_1, ..., V_k\}$ *of* $V$ *where* $V_i$ *denotes the (possibly empty) set of vertices assigned color* $i$. *The* $V_i$*'s are the color classes.*

⋆ *A proper* $k$*-coloring of a graph* $G = (V, E)$ *is a* $k$*-coloring in which each color class is a* **stable set**.

2.

⋆ *(A graph* $G$ *is* 1*-colorable) if and only if (*$G$ *is empty).*

⋆ *(A graph* $G$ *is* 2*-colorable) if and only if (*$G$ *is bipartite).*
**In particular case**, *a tree is* 2*-colorable (tree is a bipartite graph).*

3.

*We are only concerned with proper colorings. So, we refer to a proper coloring as a "***coloring***" and to a proper* $k$*-coloring as a "*$k$**-coloring***".*

4.

(a) *If* $H$ *a subgraph (non null) of a graph* $G$, *then* $\chi(H) \leq \chi(G)$.
(b) *For each graph* $G$, $\chi(G) \geq \omega(G)$.
(c) *Let* $X_1, ..., X_k$ *be the connected components of a disconnected graph* $G$, *and let* $G_i = G[X_i]$; *for* $i \in \{1, ..., k\}$. *Clearly,* $\chi(G) = max\{\chi(G_i); 1 \leq i \leq k\}$.
(d) *Given a graph* $G$, $\chi(G)$ *is the smallest positive integer* $k$ *for which there is a partition* $\{V_1, ..., V_k\}$ *of* $V$, *on* $k$ *nonempty stable sets of* $G$.

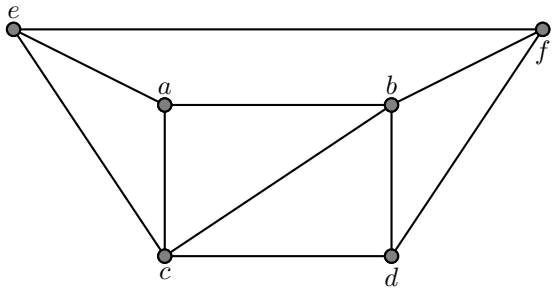
**Example 1.3**

1. **Example 1:**

(a) $\chi(K_p) = p$; $\forall p \geq 1$.
(b) $\chi(D_p) = 1$; $\forall p \geq 1$.
(c) $\chi(C_3) = 3$ *and* $\chi(C_4) = 2$.
**In general:**$\chi(C_{2p+1}) = 3$; $\forall p \geq 1$, *and* $\chi(C_{2p}) = 2$; $\forall p \geq 2$.
(d) $\chi(T) = 2$; *pour tout tree* $T$.

2. **Example 2:**

*Find the chromatic number of following graph*
$G = (\{a, b, c, d, e, f\}, \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, f\}, \{c, d\}, \{c, e\}, \{d, f\}, \{f, e\}\})$.




$G$

*Figure* $(a)$

$\star$ *As* $\{a, b, c\}$ *is a clique of* $G$*, then* $\chi(G) \geq 3$.
$\star$ *Consider the subsets* $V_1$*,* $V_2$*, and* $V_3$*, of* $V(G) = \{a, b, c, d, e, f\}$*, defined by:* $V_1 = \{a, d\}$*,* $V_2 = \{b, e\}$*, and* $V_3 = \{c, f\}$.

*Clearly,* $\{V_1, V_2, V_3\}$ *is a partition of* $V(G)$*, on non empty stable sets of* $G$*. So,* $\chi(G) \leq 3$.
*Thus,* $\chi(G) = 3$*, and a* 3*-coloring of* $G$ *is giving by: For* $i \in \{1, 2, 3\}$*, the vertices in* $V_i$ *are assigned color* $i$.

3. **Application 1: (The Scheduling Problem)**

*The students at a certain university have annual examinations in all courses they take. Naturally,examinations in different courses cannot be held concurrently if the courses have students in common. How can all the examinations be organized in as few parallel sessions as possible?*

**Solution**
*Consider the graph* $G$ *whose vertices are the courses, two courses being joined by an edge if they give rise to a conflict.*
*Thus, stable sets of* $G$ *correspond to conflict-free groups of courses.*
*It ensues that the required minimum number of parallel sessions is* $\chi(G)$.

4. **Application 2: (Chemical Storage)**

*A company manufactures* $n$ *chemicals* $C_1, C_2, ..., C_n$*. Certain pairs of these chemicals are incompatible and would course explosions if brought into contact with each other. As a precautionary measure, the company wishes to divide its warehouse into compartments, and store incompatible chemicals in different compartments.*
*What is the least number of compartments into which the warehouse should be partitioned?*

**Solution**
*Consider the graph* $G$ *whose vertices are* $C_1, C_2, ..., C_n$*, two chemicals being joined by an edge if they are incompatible.*
*Clearly, the least number of compartments into which the warehouse should be partitioned in equal to* $\chi(G)$.

# 2 Some results

**Theorem 2.1**

*Given a graph $G$ we have:* $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.

**Proof.**

$\rightarrow$ We know that: $\omega(G) \leq \chi(G)$

$\rightarrow$ To prove that: $\chi(G) \leq \Delta(G) + 1$, consider an integer $k > \Delta(G) + 1$ and let $\mathcal{C}_1$ be a $k$-coloring of $G$. The colors are: $1, 2, ..., k$.

**N. B: It sufices to prove that there is a $(k-1)$-coloring $\mathcal{C}_2$ of $G$.**

• Consider a vertex $v$ of $G$ assigned color $k$.

As, $|N_G(v)| \leq \Delta(G) < k - 1$, then there is a color $i \in \{1, 2, ..., k-1\}$ such that $i$ is not used by the vertices in $N_G(v)$. Then, we may assign the color $i$ for the vertex $v$, and we obtain a coloring of $G$. We may use this procedure to all vertices assigned color $k$ by the $k$-coloring $\mathcal{C}_1$. Thus, we obtain a $(k-1)$-coloring $\mathcal{C}_2$ of $G$.

**Remark 2.2**

$\star$ *This result is better possible, because: For* $G \in \{K_p, C_{2p+1}\}$, $\chi(G) = \Delta(G) + 1$ *(where* $p \geq 1$*).*

$\star$ *But, we have the following result of Brook; which will state without proof:*

**Theorem 2.3** *(***Brook's Theorem***)*

*If $G$ is a connected graph and it is neither an odd cycle nor a complete graph, then:* $\chi(G) \leq \Delta(G)$.

**Remark 2.4** *(***"The four color problem"***)*

*Consider a map divided into countries. We assign a color to each country, so that adjacent countries (i.e. countries sharing some common boundary) are assigned different colors.*
*What is the least number of colors required to color all countries in the map.*

**Problem**

*No map would require more than four colors* (1852).

**N. B** *This problem was solved by Appel and Haken, in* (1976). *But the proof uses computer programs for many cases.*
*There is no purely mathematical proof.!!!*
*This result is then:*

**Theorem: (The four color Theorem)**

*If $G$ is a planar graph, then:* $\chi(G) \leq 4$.

Here we prove the five color Theorem.

**Theorem 2.5 The five color Theorem**

*If $G$ is a planar graph, then:* $\chi(G) \leq 5$.

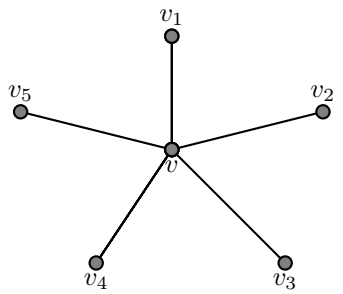
**Proof.**

By induction on $p = |V(G)| \geq 1$.

$\star$ For $p = 1$, the result is trivial.

$\star$ Let $k \geq 2$ be an integer and assume that all planar graph with $p = k-1$ vertices have chromatics number at most 5, and let $G$ be a planar graph of order $k$. We know that $\delta(G) \leq 5$. Let $v$ be a vertex with $d_G(v) = \delta(G)$. By induction hypothesis, the induced subgraph $G - v$ has a 5-coloring. Denoting the colors by: $1, 2, 3, 4$, and 5. If one of these colors is not used in coloring the vertices adjacent with $v$, then we may assign that color to $v$, and we obtain a 5-coloring of $G$.

Thus, we may assume that: $d_G(v) = 5$ and that all the five colors are used by the vertices adjacent with $v$. Let us draw $G$ as plane graph, and suppose $v_1, v_2, v_3, v_4, v_5$ are the 5 vertices adjacent with $v$, arranged cyclically about $v$, and suppose $v_i$ has been colored by $i$; $(\forall i \in \{1, 2, 3, 4, 5\})$.




**N. B:** We will show that it is possible to recolor certain vertices of $G - v$, **including a vertex adjacent to** $v$, so a color becomes available for $v$.

$\rightarrow$ Consider the subgraph $H$ of $G$ defined by: $H = G[V_{1,3}]$ where $V_{1,3}$ denotes the set of vertices of $G - v$ which have been colored 1 or 3.

**Case 1:** If $v_1$ and $v_3$ are **different connected component** $X_1$ and $X_3$, respectively, of $H$.

We interchange the two colors of the vertices of $X_1$, and then we obtain another 5-coloring of $G - v$. However, this 5-coloring of $G - v$ assigns the color 3 to both $v_1$ and $v_3$. Thus, we may now assign the color 1 to the vertex $v$, producing a 5-coloring of $G$. So, $\chi(G) \leq 5$.

**Case 2:** If $v_1$ and $v_3$ are in the **same connected components** $X$ of $H$.

In this case, there is a $v_1v_3$- path $P$ in $H$. Thus, $P$ is also a $v_1v_3$- path $P$ of $G - v$, whose vertices are all colored 1 or 3. We add to $P$, the path $(v_3, v, v_1)$ and we obtain a cycle in $G$ which either encloses $v_2$ or encloses $v_4$ and $v_5$:

Thus, as $G$ drawn as a plane graph, there is no $v_2v_4$-path in $G - v$ whose vertices are colored 2 or 4. So, in the subgraph $H' = G[V_{2,4}]$ where $V_{2,4}$ denotes the set of vertices of $G - v$ which have been colored 2 or 4, $v_2$ and $v_4$ are in the **different connected components** $X_2$ and $X_4$, respectively, of $H'$. As before, we interchange the two colors of the vertices in $X_2$, and we obtain a new 5-coloring of $G - v$ in which both $v_2$ and $v_4$ are colored 4. Hence, we may now assign the color 2 to the vertex $v$, and we obtain a 5-coloring of $G$. Thus, $\chi(G) \leq 5$.

# 3 Exercises

**Conventions.** All graphs are finite, simple, and undirected. For each chromatic number requested, establish a lower bound and construct a proper coloring attaining it. Graphs are specified explicitly so that they can be drawn. The notation $G + uv$ means that the missing edge $uv$ is added to $G$.

## Level 1: Basic examples

**Exercise 3.1 (A triangle with pendant vertices)** *Let*

$$V(G) = \{a, b, c, x, y, z\}, \qquad E(G) = \{ab, bc, ca, ax, by, cz\}.$$

1. *Draw $G$.*
2. *Determine $\omega(G)$, the largest size of a clique.*
3. *Determine $\chi(G)$.*
4. *Explain why adding pendant vertices to a graph containing an edge does not change its chromatic number.*

**Exercise 3.2 (A cycle with one chord)** *Start with the cycle $C_6 = v_1v_2v_3v_4v_5v_6v_1$. Define*

$$G_1 = C_6 + v_1v_3, \qquad G_2 = C_6 + v_1v_4.$$

1. *Draw both graphs.*
2. *Determine which of them are bipartite.*
3. *Compute $\chi(G_1)$ and $\chi(G_2)$.*
4. *Generalize: when does adding a chord to an even cycle preserve bipartiteness?*

**Exercise 3.3 (A complete graph with one edge removed)** *Let $n \geq 3$ and let $G$ be obtained from $K_n$ by deleting one edge $uv$.*

1. *Draw the example with $n = 5$.*
2. *Find a clique of order $n - 1$ in $G$.*
3. *Construct a proper coloring with $n - 1$ colors.*
4. *Deduce $\chi(G)$.*

**Exercise 3.4 (Complete multipartite graphs)** *For positive integers $n_1, \ldots, n_r$, where $r \geq 2$, the graph $K_{n_1,\ldots,n_r}$ has vertex partition*

$$V = V_1 \dot{\cup} \cdots \dot{\cup} V_r, \qquad |V_i| = n_i.$$

*Two vertices are adjacent if and only if they belong to different parts.*

1. *Draw $K_{2,2,1}$ and determine its chromatic number.*
2. *Determine $\chi(K_{n_1,\ldots,n_r})$.*
3. *Determine its independence number and clique number.*

## Level 2: Structural arguments

**Exercise 3.5 (The Petersen graph)** *Let $P$ have vertices $u_0, \dots, u_4, v_0, \dots, v_4$ and edges*

$$u_i u_{i+1}, \qquad u_i v_i, \qquad v_i v_{i+2} \qquad (0 \le i \le 4),$$

*where subscripts are read modulo 5.*

1. *Draw $P$ using an outer pentagon and an inner star.*
2. *Show that $\chi(P) \ge 3$.*
3. *Construct an explicit proper 3-coloring.*
4. *Deduce $\chi(P)$.*

**Exercise 3.6 (Joining two graphs)** *For graphs $G, H$ on disjoint vertex sets, the* join *$G \vee H$ is obtained from their disjoint union by adding every edge between $V(G)$ and $V(H)$.*

1. *Prove that $\chi(G \vee H) = \chi(G) + \chi(H)$.*
2. *Draw $K_2 \vee C_5$ and compute its chromatic number.*
3. *Compute $\chi(C_5 \vee C_7)$.*
4. *The wheel $W_n$, for $n \ge 4$, is obtained from $C_{n-1}$ by adding a vertex adjacent to every cycle vertex. Express $W_n$ as a join and determine $\chi(W_n)$.*

**Exercise 3.7 (Gluing along a clique)** *Suppose $V(G_1) \cap V(G_2) = S$, where $S$ induces a complete graph in both graphs. Let $G = G_1 \cup G_2$, with no additional edges between $V(G_1) \setminus S$ and $V(G_2) \setminus S$.*

1. *Prove that $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$.*
2. *Draw two copies of $K_4$ sharing exactly one edge, and determine the chromatic number of their union.*
3. *Repeat for a copy of $K_4$ and a copy of $C_5$ sharing exactly one edge.*

**Exercise 3.8 (Deleting a matching from a complete graph)** *Let $n \ge 2$. Let $M$ be a matching of size $r$ in $K_n$, where $0 \le r \le \lfloor n/2 \rfloor$, and set $G = K_n - M$. A matching is a set of edges with pairwise disjoint endpoints.*

1. *Show that every independent set of $G$ has at most two vertices.*
2. *Show that every independent set of size two consists of the endpoints of an edge of $M$.*
3. *Prove that $\chi(G) = n - r$.*
4. *Draw the graph obtained from $K_6$ by deleting the edges $12, 34, 56$, and determine its chromatic number.*
5. *Determine the chromatic number of $K_8$ with a perfect matching removed.*

## Level 3: More challenging examples

**Exercise 3.9 (The complement of a cycle)** *Let $n \geq 4$. In $\overline{C_n}$, distinct vertices are adjacent exactly when they are not adjacent in $C_n$.*

1. *Draw $\overline{C_6}$ and $\overline{C_7}$.*
2. *Prove that $\alpha(\overline{C_n}) = 2$.*
3. *Deduce $\chi(\overline{C_n}) \geq \lceil n/2 \rceil$.*
4. *Partition the cycle vertices into adjacent pairs and, if necessary, one singleton. Use this partition to prove*
$$\chi(\overline{C_n}) = \left\lceil \frac{n}{2} \right\rceil .$$
5. *Compute $\chi(\overline{C_7})$ and $\omega(\overline{C_7})$.*

**Exercise 3.10 (Squares of cycles)** *For $n \geq 3$, the square $C_n^2$ has the same vertices as $C_n$; distinct vertices are adjacent when their distance in $C_n$ is at most two.*

1. *Draw $C_5^2$, $C_6^2$, and $C_7^2$.*
2. *Determine their chromatic numbers.*
3. *For $n \geq 6$, prove that $\alpha(C_n^2) \leq \lfloor n/3 \rfloor$.*
4. *For $n \geq 6$, prove that $C_n^2$ is 3-colorable if and only if 3 divides $n$.*

Hint. *Every three consecutive vertices form a triangle. In a 3-coloring compare the colors of $v_i$ and $v_{i+3}$, with subscripts modulo $n$.*

**Exercise 3.11 (The Mycielski construction)** *Let $G$ be a nonempty graph with vertices $v_1, \ldots, v_n$. Construct $M(G)$ by adding vertices $u_1, \ldots, u_n, w$:*

- *keep all edges of $G$;*
- *for every edge $v_iv_j$ of $G$, add $u_iv_j$ and $u_jv_i$;*
- *join $w$ to every $u_i$;*
- *add no other edges.*

1. *Show that $\chi(M(G)) \leq \chi(G) + 1$.*
2. *Suppose that $M(G)$ has a proper coloring with $q$ colors and $w$ has color $a$. For each original vertex $v_i$ colored $a$, replace its color by that of $u_i$. Prove that this gives a proper coloring of $G$ using at most $q - 1$ colors.*
3. *Deduce $\chi(M(G)) = \chi(G) + 1$.*
4. *Show that if $G$ is triangle-free, then so is $M(G)$.*
5. *Starting from $C_5$, describe and draw a triangle-free graph of order 11 with chromatic number 4.*

**Exercise 3.12 (Greedy coloring can perform badly)** *For $n \geq 2$, let $H_n$ have bipartition*

$$A = \{a_1, \ldots, a_n\}, \qquad B = \{b_1, \ldots, b_n\},$$

*with $a_i b_j \in E(H_n)$ if and only if $i \neq j$. There are no edges inside either part.*

1. *Draw $H_3$ and determine its chromatic number.*
2. *Determine $\chi(H_n)$ for every $n \geq 2$.*
3. *Apply the greedy algorithm, using the smallest available positive integer, in the order*

   $$a_1, b_1, a_2, b_2, \ldots, a_n, b_n.$$

   *Prove that $a_i$ and $b_i$ both receive color $i$.*
4. *Give an ordering for which greedy coloring uses only two colors.*
5. *Explain why there is no universal constant bounding the ratio between the number of colors used by greedy coloring in an arbitrary ordering and the chromatic number.*