Chapter 5
One and Two-Sample Tests of Hypotheses

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1

Introduction

Introduction

Consider a population with some unknown parameter Θ. We are interested in testing (confirming or denying) some conjectures about Θ. For example, we might be interested in testing the conjecture that

$$\Theta = \Theta_0 \qquad \text{where } \Theta_0 \text{ is a given value}$$

Definition

A **statistical hypothesis** is an assertion or conjecture concerning one or more populations.

2

The Null and Alternative Hypotheses

Null and alternative hypotheses

We usually test the null hypothesis

$$H_0: \Theta = \Theta_0$$

against one of the following **alternative hypotheses** (Alternative Hypothesis or Research Hypothesis)

$$H_1: \begin{cases} \Theta \neq \Theta_0 \\ \Theta > \Theta_0 \\ \Theta < \Theta_0 \end{cases}$$

Possible situations in testing a statistical hypothesis

	H_0 is true	H_0 is false
Accepting H_0	**Correct decision**	**Type II error (β)**
Rejecting H_0	**Type I error (α)**	**Correct decision**

Type I and type II errors

Definition of Type I errors

Rejection of the null hypothesis (H_0) when it is true is called a **type I error**.

The probability of committing a type I error, also called the **level of significance**, is denoted by α.

$$\Pr(\text{type I error}) = \Pr(\text{Rejecting } H_0 \mid H_0 \text{ is true}) = \alpha$$

Definition Type II error

Nonrejection of the null hypothesis (H_0) when it is false is called a **type II error**.

The probability of committing a type II error, denoted by β, is impossible to compute unless we have a specific alternative hypothesis.

$$\Pr(\text{type II error}) = \Pr(\text{Accepting } H_0 \mid H_0 \text{ is false}) = \beta$$

3

Single Sample: Tests Concerning a Single Mean

Test statistic Z when σ is known · T when σ is unknown

Two-sided test on the mean (σ known)

Z

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance $\sigma^2 > 0$. Consider first the hypothesis

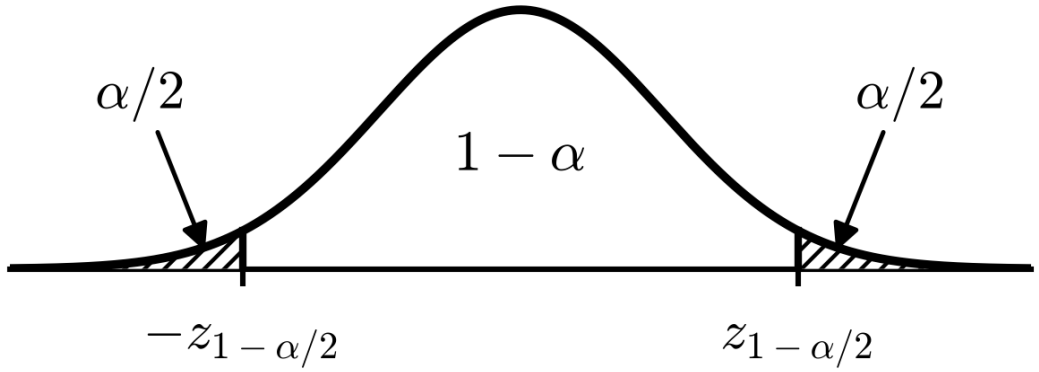
$$H_0: \mu = \mu_0 \quad \text{vs} \quad H_1: \mu \neq \mu_0$$

Test statistic (T.S.)

$$Z_{T.S} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

and

$$\Pr\left(-z_{1-\alpha/2} \leq \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq z_{1-\alpha/2}\right) = 1 - \alpha$$



Two-sided alternative

Shaded area = rejection region

Test procedure (variance known)

Z

Theorem

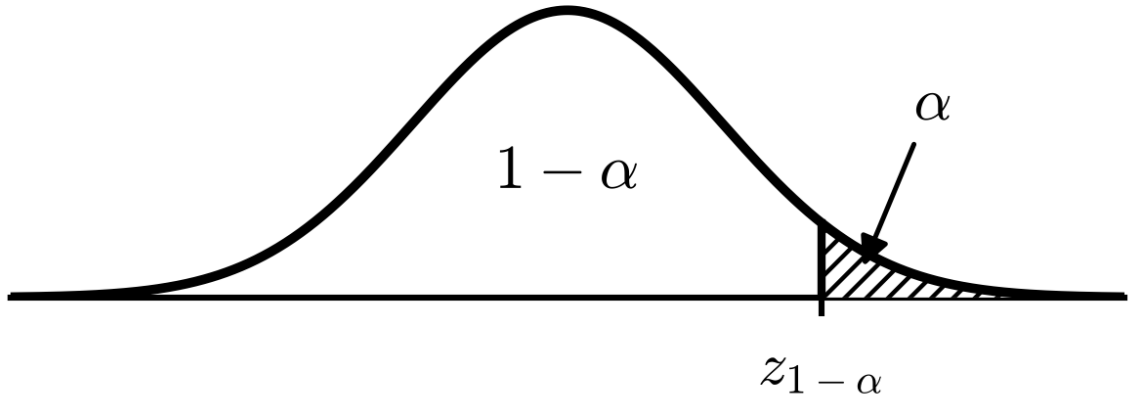
Do not reject H_0 if $-z_{1-\alpha/2} < Z_{T.S} < z_{1-\alpha/2}$.

Rejection of H_0, of course, implies acceptance of the alternative hypothesis $\mu \neq \mu_0$. With this definition of the **critical region** it should be clear that there will be probability α of rejecting H_0 (falling into the critical region) when indeed $\mu = \mu_0$.

One-sided test on the mean : H_1 has the sign >

Z

$$H_0\colon \mu = \mu_0 \quad \text{vs} \quad H_1\colon \mu > \mu_0$$



One-sided alternative

Shaded area = rejection region

Test statistic (T.S.)

$$Z_{T.S} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

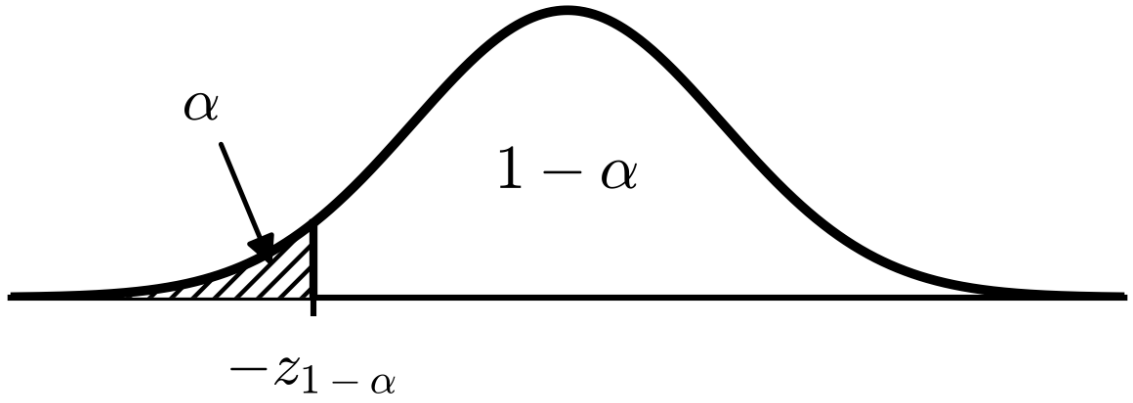
and

$$\Pr\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq z_{1-\alpha}\right) = 1 - \alpha$$

One-sided test on the mean : H_1 has the sign <

Z

$$H_0\colon \mu = \mu_0 \quad \text{vs} \quad H_1\colon \mu < \mu_0$$



One-sided alternative

Shaded area = rejection region

Test statistic (T.S.)

$$Z_{T.S} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

and

$$\Pr\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq -z_{1-\alpha}\right) = 1 - \alpha$$

Conclusion: test procedure

Z

First Case: σ^2 is known; Normal or Non-Normal Distribution ($n \geq 30$)

Test statistic for all three cases

$$Z_{T.S} = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

Two-sided test

H_0: $\mu = \mu_0$

H_1: $\mu \neq \mu_0$

Reject H_0 (and accept H_1) at level α if

$Z_{T.S} > z_{1-\alpha/2}$ or $Z_{T.S} < -z_{1-\alpha/2}$

One-sided test

H_0: $\mu = \mu_0$

H_1: $\mu > \mu_0$

Reject H_0 (and accept H_1) at level α if

$Z_{T.S} > z_{1-\alpha}$

One-sided test

H_0: $\mu = \mu_0$

H_1: $\mu < \mu_0$

Reject H_0 (and accept H_1) at level α if

$Z_{T.S} < -z_{1-\alpha}$

Note z with subscript 1−α has area 1−α to its left. Example z(0.95) = 1.645

P

How to Find the P-value

Definition, decision rule and the three cases

What is the P-value

P

Definition

A **P-value** is the **lowest level** (of significance) at which the observed value of the test statistic is **significant**.

Decision rule

P-value < α

Reject H_0 and accept H_1

P-value ≥ α

Do not reject H_0

A small P-value means strong evidence against H_0

P-value summary

P

Case	Sign in H_1	P-value	Where is the area
Right-tailed	$>$	$P(Z > z_{T.S})$	Right of $z_{T.S}$
Left-tailed	$<$	$P(Z < z_{T.S})$	Left of $z_{T.S}$
Two-tailed	$\neq$	$2\,P(Z > \lvert z_{T.S} \rvert)$	Both tails

Tip The same three rules work for the t, χ^2 and F tests. Their tables only give a few values, so we often write the P-value as an approximate value (for example P-value ≈ 0.06).

Example 1

Ex

Problem

A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution

Population	Sample	α
$\sigma = 8.9$	$n = 100, \quad \bar{x} = 71.8$	**0.05**

Example 1 Solution (steps 1 ,2 and 3)

Z

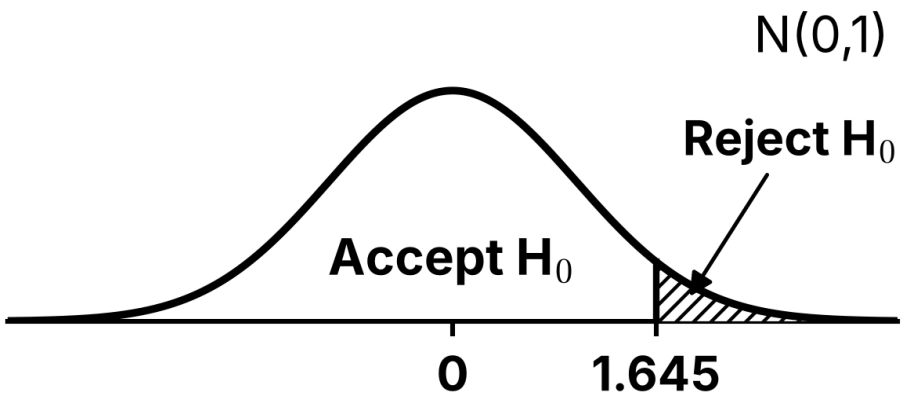
1 Hypotheses

H_0: $\mu = 70$ vs H_1: $\mu > 70$, $\mu_0 = 70$

2 Test statistic (Non-normal population, σ is known and n ≥ 30)

$$Z_{T.S.} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{71.8 - 70}{8.9/\sqrt{100}} = 2.02 \sim N(0,1)$$

3 Rejection region



Critical value

$$Z_{1-\alpha} = Z_{0.95} = 1.645$$

Example 1 Solution (steps 4)

Z

4 Decision

AS $Z_{T.S.} = 2.02 > 1.645$

we reject H_0 and conclude that the mean life span today is greater than 70 years.

The P-value corresponding to z = 2.02 is given by the area on the right under the density of standard normal distribution. Using Table A.3, we have

$$\text{P}-\text{value} = P(Z > 2.02) = 1 - P(Z < 2.02) = 0.0217$$

As $\text{P}-\text{value} = 0.0217 < \alpha = 0.05$ so we have the same decision

Example 2

Ex

Problem

A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.01 level of significance.

Solution

Population	Sample	α
$\mu = 8, \quad \sigma = 0.5$	$n = 50, \quad \bar{x} = 7.8$	**0.01**

Example 2 Solution (steps 1, 2 and 3)

Z

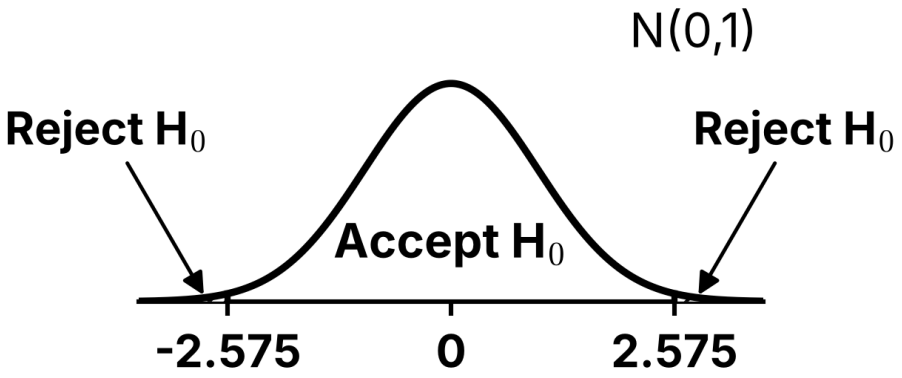
1 Hypotheses

H_0: $\mu = 8$ vs H_1: $\mu \neq 8$, $\mu_0 = 8$

2 Test statistic (Non-normal population, σ is known and n ≥ 30)

$$Z_{T.S.} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{7.8 - 8}{0.5/\sqrt{50}} = -2.83 \sim N(0,1)$$

3 Rejection region



Critical values

$$Z_{\alpha/2} = -Z_{1-\alpha/2} = -2.575$$

$$Z_{1-\alpha/2} = Z_{0.995} = 2.575$$

Example 2 Solution (step 4)

Z

4 Decision

As $Z_{T.S.} = -2.83 < -2.575$

we reject H_0 and conclude that the average breaking strength is not equal to 8 but is, in fact, less than 8 kilograms.

Since the test in this example is two tailed, the desired P-value is twice the area to the left of z = −2.83. Using the standard normal table, we have

$$\text{P}-\text{value} = 2P(Z > |-2.83|) = 2P(Z > 2.83) = 0.0046$$

As $\text{P}-\text{value} = 0.0046 < \alpha = 0.01$ so we have the same decision

which allows us to reject the null hypothesis that μ = 8 kilograms at a level of significance smaller than 0.01.

Tests on one mean (σ unknown)

Test statistic

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

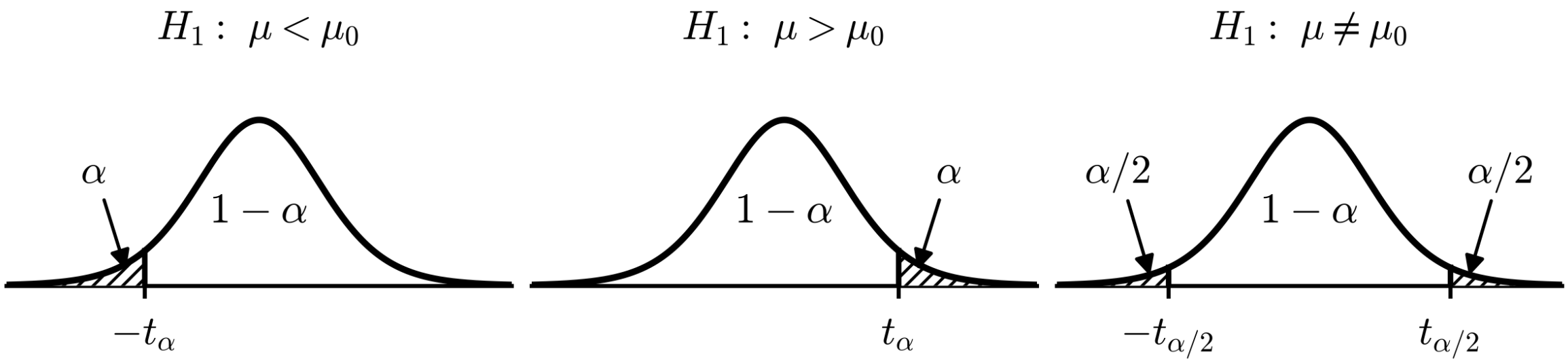
with n – 1 degrees of freedom

When σ is **unknown** we use the sample standard deviation S in place of σ, and the test statistic follows a t distribution.

We reject H_0 at level α when

$H_1: \mu \neq \mu_0$	$T > t_{\alpha/2,\, n-1}$ or $T < -t_{\alpha/2,\, n-1}$
$H_1: \mu > \mu_0$	$T > t_{\alpha,\, n-1}$
$H_1: \mu < \mu_0$	$T < -t_{\alpha,\, n-1}$

Rejection regions for the t test



Shaded area = rejection region (R.R.). The rest is the acceptance region (A.R.).

Note In the t, χ^2 and F tables the subscript is the area to the right. For example t(0.05, 11) = 1.796 has area 0.05 to its right.

Example 3

Ex

Problem

The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of 46 kilowatt hours per year. If a random sample of 12 homes indicates that vacuum cleaners use an average of 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours, does this suggest at the 0.05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually? Assume the population of kilowatt hours to be normal.

Solution

Population	Sample	α
$\mu = 46$, Normal	$n = 12$, $\bar{x} = 42$, $s = 11.9$	**0.05**

Example 3 Solution (steps 1, 2 and 3)

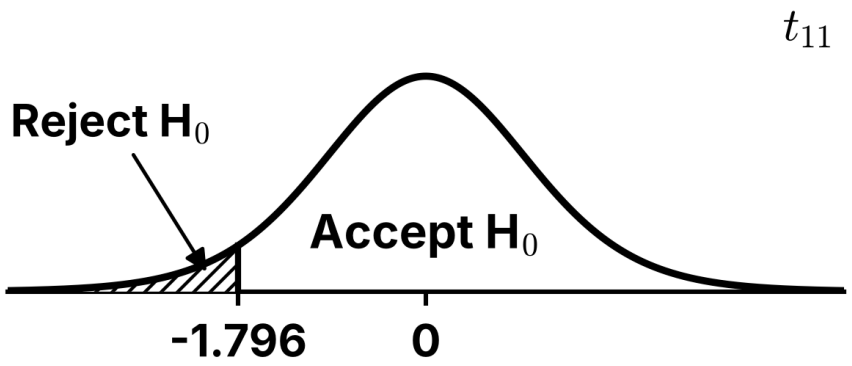
1 Hypotheses

H_0: $\mu = 46$ vs H_1: $\mu < 46$, $\mu_0 = 46$

2 Test statistic (Normal population, σ is unknown and n < 30)

$$T_{T.S.} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16 \sim t_{11}$$

3 Rejection region



Critical value

$t_{1-\alpha,\,11} = -t_{\alpha,\,11} = -t_{0.05,\,11} = -1.796$

with 11 degrees of freedom

Example 3 Solution (step 4)

T

4 Decision

As $T_{T.S.} = -1.16 > -1.796$

we do not reject H_0 and conclude that the average number of kilowatt hours used annually by home vacuum cleaners is not significantly less than 46.

The P-value is the area to the left of t = −1.16 under the t distribution with 11 degrees of freedom. Using Table A.4, we have

$$\text{P-value} = P(T < -1.16) = P(T > 1.16) \approx 0.135$$

As $\text{P-value} \approx 0.135 > \alpha = 0.05$ so we have the same decision

4

Two Samples: Tests on Two Means

Z when variances are known · T when variances are unknown or data are paired

Two means with known variances

Z

The two-sided hypothesis on two means can be written generally as H_0: $\mu_1 = \mu_2$. For σ_1 and σ_2 known the test statistic is given by

$$Z_{T.S} = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

Reject H_0 in favor of H_1: $\mu_1 \neq \mu_2$ if

$$Z_{T.S} > z_{1-\alpha/2} \quad \text{or} \quad Z_{T.S} < -z_{1-\alpha/2}$$

One-sided alternatives

One-tailed critical regions are used in the case of the one-sided alternatives. For H_1: $\mu_1 > \mu_2$ the signal favoring H_1 comes from large values of z. Thus the upper-tailed critical region applies.

Unknown but equal variances: pooled t-test

If we assume that both distributions are normal and that $\sigma_1 = \sigma_2 = \sigma$, the two-sample pooled t-test may be used.

Test statistic

$$T = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{1/n_1 + 1/n_2}} \quad \text{where} \quad S_p^2 = \frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

Degrees of freedom $n_1 + n_2 - 2$

Pooled t-test: rejection rules

We reject H_0 at significance level α when

$H_1: \mu_1 \neq \mu_2$	$T > t_{\alpha/2,\, n_1+n_2-2}$ or $T < -t_{\alpha/2,\, n_1+n_2-2}$
$H_1: \mu_1 > \mu_2$	$T > t_{\alpha,\, n_1+n_2-2}$
$H_1: \mu_1 < \mu_2$	$T < -t_{\alpha,\, n_1+n_2-2}$

Example 4

Ex

Problem

An experiment was performed to compare the abrasive wear of two materials. Twelve pieces of material 1 were tested and ten pieces of material 2 were similarly tested. The samples of material 1 gave an average (coded) wear of 85 units with a sample standard deviation of 4, while the samples of material 2 gave an average of 83 with a sample standard deviation of 5. Can we conclude at the 0.05 level of significance that the abrasive wear of material 1 exceeds that of material 2? Assume the populations to be approximately normal with equal variances.

Example 4 Solution

Solution

Let μ_1 and μ_2 represent the population means of the abrasive wear for material 1 and material 2, respectively.

Group	Population	Sample	α
(1) Material 1	Normal, $\sigma_1 = ?$	$n_1 = 12$, $\bar{x}_1 = 85$, $s_1 = 4$	0.05
(2) Material 2	Normal, $\sigma_2 = ?$	$n_2 = 10$, $\bar{x}_2 = 83$, $s_2 = 5$	

Example 4 Solution (steps 1 and 2)

1 Hypotheses

$H_0: \mu_1 = \mu_2 \quad \text{vs} \quad H_1: \mu_1 > \mu_2$

2 Test statistic (Normal populations, $\sigma_1 = \sigma_2 = \sigma$ unknown, $n_1, n_2 < 30$)

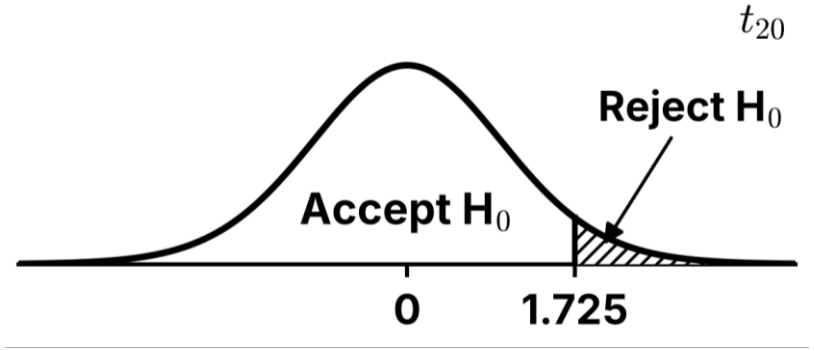
$$T_{T.S.} = \frac{\bar{x}_1 - \bar{x}_2}{S_p\sqrt{1/n_1 + 1/n_2}} \sim t_{\alpha,\, n_1+n_2-2}$$

$$S_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} = \sqrt{\frac{11(4)^2 + 9(5)^2}{12+10-2}} = 4.478$$

$$T_{T.S.} = \frac{85-83}{4.479\sqrt{1/12+1/10}} = 1.04$$

Example 4 Solution (step 3 and 4)

3 Rejection region



Critical value

$t_{\alpha,\, n_1+n_2-2} = t_{0.05,\, 20} = 1.725$

with $\nu = 20$ degrees of freedom

4 Decision

As $T_{T.S.} = 1.04 < 1.725$

we do not reject H_0. We are unable to conclude that the abrasive wear of material 1 exceeds that of material 2.

The P-value is the area to the right of t = 1.04 under the t distribution with 20 degrees of freedom. Using Table A.4, we have

$\text{P}-\text{value} = P(T > 1.04) \approx 0.16$

As $\text{P}-\text{value} \approx 0.16 > \alpha = 0.05$, so we have the same decision

Paired observations for testing on two means

Testing of two means can be accomplished when data are in the form of paired observations, as discussed in Chapter 4. The statistical test for two means μ_1 and μ_2 with paired observations is based on the random variable

$$T = \frac{\bar{D} - \mu_D}{S_D/\sqrt{n}}$$

where $\bar{D}$ and S_D are the sample mean and standard deviation of the differences of the observations in the experimental units.

Paired observations: key ideas

1. The observations from each population are assumed **normal**

2. The two-sample problem is essentially reduced to a **one-sample problem** by using the differences $d_1, d_2, \ldots, d_n$

3. Critical regions are constructed using the t-distribution with **n – 1 degrees of freedom.**

Example 5 Blood sample data

Ex

Problem

A study was conducted to examine the influence of the drug succinylcholine on the circulation levels of androgens in the blood. Blood samples were taken from wild, free-ranging deer immediately after they had received an intramuscular injection of succinylcholine. A second blood sample was obtained from each deer 30 minutes after the first sample. The androgen levels (ng/mL) for 15 deer are given in the table on the next slide. Assuming that both populations are normally distributed, test at the 0.05 level of significance whether the androgen concentrations are altered after 30 minutes.

Example 5 Androgen data (ng/mL)

Ex

Deer	At injection	After 30 min	d_i
1	2.76	7.02	4.26
2	5.18	3.10	−2.08
3	2.68	5.44	2.76
4	3.05	3.99	0.94
5	4.10	5.21	1.11
6	7.05	10.26	3.21
7	6.60	13.91	7.31
8	4.79	18.53	13.74

Deer	At injection	After 30 min	d_i
9	7.39	7.91	0.52
10	7.30	4.85	−2.45
11	11.78	11.10	−0.68
12	3.90	3.74	−0.16
13	26.00	94.03	68.03
14	67.48	94.03	26.55
15	17.04	41.70	24.66

$$\bar{D} = 9.848, \qquad S_D = 18.474$$

Example 5 Solution (steps 1 and 2)

T

Solution

Let μ_1 and μ_2 be the average androgen concentration at the time of injection and 30 minutes later, respectively. The data are on the next slide.

1 Hypotheses

$$H_0\colon\ \mu_D = 0\ \ (\mu_1 = \mu_2) \quad \text{vs} \quad H_1\colon\ \mu_D \neq 0\ \ (\mu_1 \neq \mu_2)$$

We use the not equal sign because the question says altered (altered means changed)

2 Test statistic

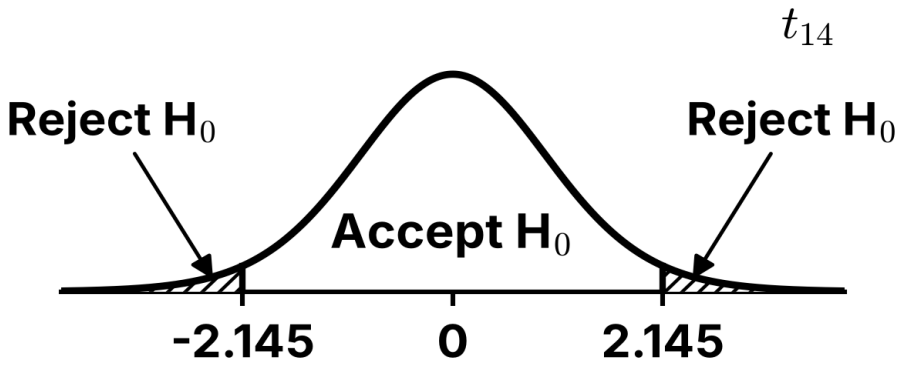
Find D, the difference between the two readings, then use the calculator

$$\bar{D} = 9.848, \qquad S_D = 18.474$$

$$T_{T.S.} = \frac{\bar{D}}{S_D/\sqrt{n}} = \frac{9.848}{18.474/\sqrt{15}} = 2.06$$

Example 5 Solution (step 3)

3 Rejection region



Critical value

$$\pm t_{\alpha/2,\,14} = \pm t_{0.025,\,14} = \pm 2.145$$

with $\nu = 14$ degrees of freedom

Example 5 Solution (step 4)

T

4 Decision

$$-2.145 < T_{T.S.} = 2.06 < 2.145$$

we do not reject H_0. The t-statistic is not significant at the 0.05 level, so there is no evidence that there is a difference in mean circulating levels of androgen.

Since the test is two tailed, the P-value is twice the area to the right of t = 2.06. Using Table A.4 with 14 degrees of freedom, we have

$$\text{P}-\text{value} = 2P(T > 2.06) \approx 0.06$$

As $\text{P}-\text{value} \approx 0.06 > \alpha = 0.05$, so we have the same decision

5

One Sample: Test on a Single Proportion

Test statistic Z

Test on a single proportion

Z

Tests of hypotheses concerning proportions are required in many areas. We test the null hypothesis H_0 that $p = p_0$, where p is the parameter of the binomial distribution. The alternative may be one of the usual one-sided or two-sided alternatives

$$p < p_0 \qquad p > p_0 \qquad p \neq p_0$$

If the conditions below hold, then $\hat{P}$ is approximately **normal** with mean p_0 and standard deviation σ

$$np_0 \geq 5 \quad \text{and} \quad n(1-p_0) \geq 5 \qquad\qquad \sigma_{\hat{P}} = \sqrt{p_0(1-p_0)/n}$$

The z-value for one proportion

Z

The z-value for testing $p = p_0$

$$Z_{T.S} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}}$$

Critical regions at level α

$p \neq p_0$ $\quad Z_{T.S} < -z_{1-\alpha/2}$ or $Z_{T.S} > z_{1-\alpha/2}$

$p < p_0$ $\quad Z_{T.S} < -z_{1-\alpha}$

$p > p_0$ $\quad Z_{T.S} > z_{1-\alpha}$

Example 6

Ex

Problem

A commonly prescribed drug for relieving nervous tension is believed to be only 60% effective. Experimental results with a new drug administered to a random sample of 100 adults who were suffering from nervous tension show that 70 received relief. Is this sufficient evidence to conclude that the new drug is superior to the one commonly prescribed? Use a 0.05 level of significance.

Solution

Population	Sample	α
$P = 60/100 = 0.6$	$n = 100,\ \ x = 70,\ \ \hat{p} = 70/100 = 0.7$	**0.05**

Example 6 Solution (steps 1 and 2)

Z

1 Hypotheses

H_0: $P = 0.6$ vs H_1: $P > 0.6$, $P_0 = 0.6$

We use > because the question says superior (superior means higher)

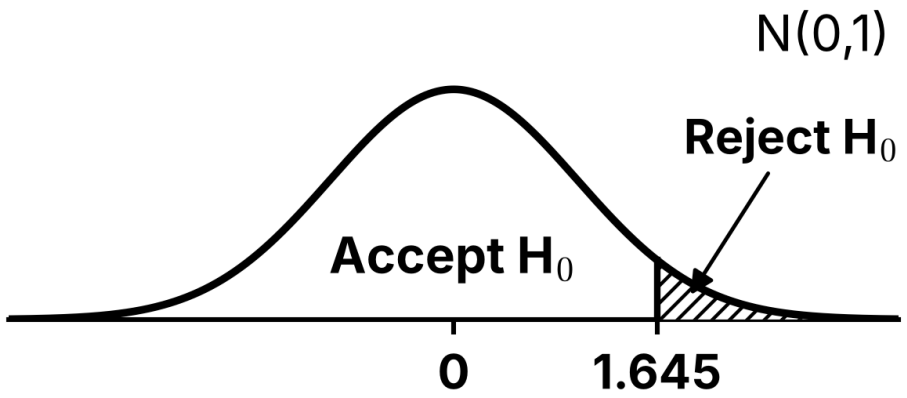
2 Test statistic (n ≥ 30)

$$Z_{T.S.} = \frac{\hat{p} - P_0}{\sqrt{P_0 Q_0 / n}} = \frac{0.7 - 0.6}{\sqrt{0.6(0.4)/100}} = 2.04 \sim N(0,1)$$

Example 6 Solution (step 3)

Z

3 Rejection region



Critical value

$$Z_{1-\alpha} = Z_{0.95} = 1.645$$

Example 6 Solution (step 4)

Z

4 Decision

As $Z_{T.S.} = 2.04 > 1.645$

we reject H_0 and conclude that the new drug is superior.

The P-value is the area to the right of z = 2.04 under the standard normal curve. Using Table A.3, we have

$\text{P}-\text{value} = P(Z > 2.04) = 0.0207$

As $\text{P}-\text{value} = 0.0207 < \alpha = 0.05$so we have the same decision

6

Two Samples: Tests on Two Proportions

Test statistic Z

Tests on two proportions

Z

We test $p_1 = p_2$ against one of the alternatives $p_1 < p_2$, $p_1 > p_2$ or $p_1 \neq p_2$. The decision is based on the random variable $\hat{P}_1 - \hat{P}_2$. When H_0: $p_1 = p_2$ (= p) is true we know that

$$Z_{T.S} = \frac{\hat{P}_1 - \hat{P}_2}{\sqrt{pq(1/n_1 + 1/n_2)}}$$

To compute Z we must estimate p and q. Under H_0 both $\hat{P}_1$ and $\hat{P}_2$ are estimators of p.

Pooled estimate and the z-value

Pooled estimate of p

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}, \qquad \hat{q} = 1 - \hat{p}$$

x_1 and x_2 are the numbers of successes in each of the two samples.

The z-value for testing $p_1 = p_2$

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}(1/n_1 + 1/n_2)}}$$

Critical regions are set up as before using the standard normal curve.

Example 7

Ex

Problem

A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. To determine if there is a significant difference in the proportions of town voters and county voters favoring the proposal, a poll is taken. If 120 of 200 town voters favor the proposal and 240 of 500 county residents favor it, would you agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters? Use an $\alpha = 0.05$ level of significance.

Example 7 Solution

Z

Solution

Let p_1 and p_2 be the true proportions of voters in the town and county, respectively, favoring the proposal.

Group	Population	Sample	α
(1) Town	—	$n_1 = 200,\ \ x_1 = 120,\ \ \hat{p}_1 = 0.6$	**0.05**
(2) County	—	$n_2 = 500,\ \ x_2 = 240,\ \ \hat{p}_2 = 0.48$	

Example 7 Solution (steps 1 and 2)

Z

1 Hypotheses

$$H_0: P_1 = P_2 \quad \text{vs} \quad H_1: P_1 > P_2 \qquad (P_1 - P_2 > 0)$$

We use > because the question says higher

2 Test statistic (n_1 and $n_2 \geq 30$)

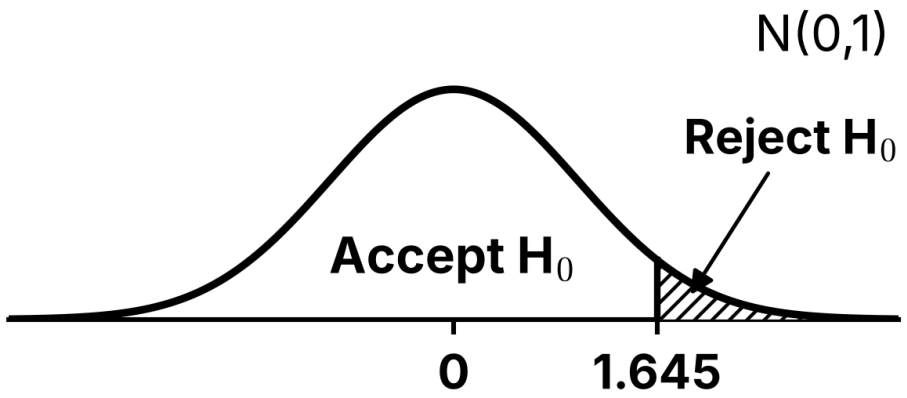
$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{120 + 240}{200 + 500} = 0.51 \qquad Z_{T.S.} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\,(1/n_1 + 1/n_2)}}$$

$$= \frac{0.6 - 0.48}{\sqrt{0.51(1 - 0.51)(1/200 + 1/500)}} = 2.9 \sim N(0,1)$$

Example 7 Solution (step 3)

Z

3 Rejection region



Critical value

$$Z_{1-\alpha} = Z_{0.95} = 1.645$$

Example 7 Solution (step 4)

Z

4 Decision

As $Z_{T.S.} = 2.9 > 1.645$

we reject H_0 and agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters.

The P-value is the area to the right of z = 2.9 under the standard normal curve. Using Table A.3, we have

$$\text{P}-\text{value} = P(Z > 2.90) = 0.0019$$

As $\text{P}-\text{value} = 0.0019 < \alpha = 0.05$ so we have the same decision

7

One- and Two-Sample Tests Concerning Variances

χ^2 for one variance · F for two variances

Test on one variance

χ^2

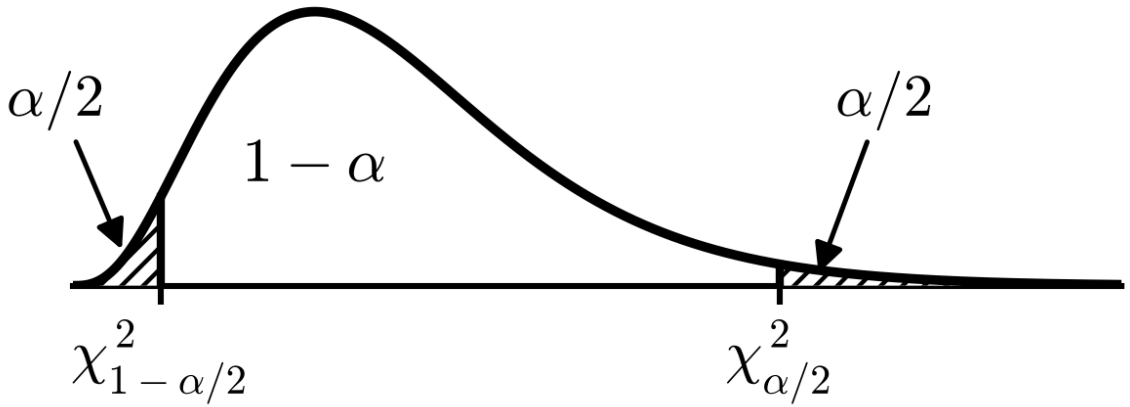
We test the null hypothesis H_0 that the population variance σ^2 equals a specified value σ_0^2 against $\sigma^2 < \sigma_0^2$, $\sigma^2 > \sigma_0^2$ or $\sigma^2 \neq \sigma_0^2$. If the population is normal the chi-squared value is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}, \qquad \nu = n - 1$$

n is the sample size, s^2 is the sample variance and σ_0^2 is the value given by H_0. If H_0 is true, χ^2 has a chi-squared distribution with $\nu = n - 1$ degrees of freedom.

Critical regions for one variance

χ^2



Two-tailed test · Shaded area = rejection region

Critical regions at level α

$\sigma^2 \neq \sigma_0^2$	$\chi^2 < \chi^2_{1-\alpha/2}$ or $\chi^2 > \chi^2_{\alpha/2}$
$\sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{1-\alpha}$
$\sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{\alpha}$

Example 8

Ex

Problem

A manufacturer of car batteries claims that the life of the company's batteries is approximately normally distributed with a standard deviation equal to 0.9 year. If a random sample of 10 of these batteries has a standard deviation of 1.2 years, do you think that $\sigma > 0.9$ year? Use a 0.05 level of significance.

Solution

Population	Sample	α
Normal, $\sigma = 0.9$	$n = 10$, $s = 1.2$	**0.05**

Example 8 Solution (steps 1, 2 and 3)

χ^2

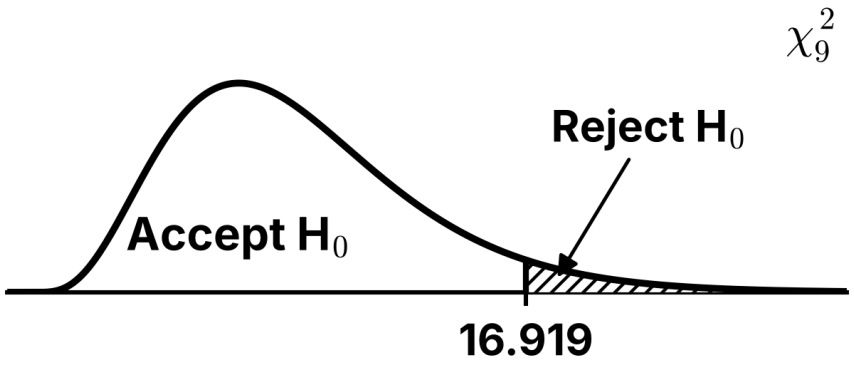
1 Hypotheses

H_0: $\sigma^2 = 0.9^2$ vs H_1: $\sigma^2 > 0.9^2$, $\sigma_0 = 0.9$

2 Test statistic (The population is normal)

$$\chi^2_{T.S.} = \frac{(n-1)s^2}{\sigma_0^2} = \frac{9(1.2)^2}{0.9^2} = 16 \sim \chi^2_9$$

3 Rejection region



Critical value

$\chi^2_{\alpha,\,9} = \chi^2_{0.05,\,9} = 16.919$

with $\nu = 9$ degrees of freedom

Example 8 Solution (step 4)

χ^2

4 Decision

As $\chi^2_{T.S.} = 16 < 16.919$

we do not reject H_0. The χ^2-statistic is not significant at the 0.05 level.

The P-value is the area to the right of $\chi^2 = 16$ under the chi-squared distribution with 9 degrees of freedom. Using Table A.5, we have

$$\text{P}-\text{value} = P(\chi^2 > 16) \approx 0.07$$

As $\text{P}-\text{value} \approx 0.07 > \alpha = 0.05$ so we have the same decision

However, based on the P-value 0.07, there is some evidence that $\sigma > 0.9$.

Test on two variances

F

We test H_0: $\sigma_1^2 = \sigma_2^2$ against $\sigma_1^2 < \sigma_2^2$, $\sigma_1^2 > \sigma_2^2$ or $\sigma_1^2 \neq \sigma_2^2$. For independent random samples of sizes n_1 and n_2 from two normal populations the f-value is the ratio

$$f = \frac{s_1^2}{s_2^2}, \qquad \nu_1 = n_1 - 1, \quad \nu_2 = n_2 - 1$$

s_1^2 and s_2^2 are the variances computed from the two samples. If H_0 is true, f is a value of the F-distribution with ν_1 and ν_2 degrees of freedom.

Critical regions for two variances

F

Critical regions of size α

$\sigma_1^2 < \sigma_2^2$ $\quad f < f_{1-\alpha}(\nu_1, \nu_2)$

$\sigma_1^2 > \sigma_2^2$ $\quad f > f_{\alpha}(\nu_1, \nu_2)$

$\sigma_1^2 \neq \sigma_2^2$ $\quad f < f_{1-\alpha/2}(\nu_1, \nu_2)$ or $f > f_{\alpha/2}(\nu_1, \nu_2)$

To find the lower value

$$f_{1-\alpha/2}(\nu_1, \nu_2) = \frac{1}{f_{\alpha/2}(\nu_2, \nu_1)}$$

Example 9

Ex

Problem

In testing for the difference in the abrasive wear of the two materials in Example 4, we assumed that the two unknown population variances were equal. Were we justified in making this assumption? Use a 0.10 level of significance.

Example 9 Solution

F

Solution

Let σ_1^2 and σ_2^2 be the population variances for the abrasive wear of material 1 and material 2, respectively.

Group	Population	Sample	α
(1) Material 1	Normal	$n_1 = 12,\ \ s_1^2 = 16$	**0.1**
(2) Material 2	Normal	$n_2 = 10,\ \ s_2^2 = 25$	

Example 9 Solution (steps 1 and 2)

F

1 Hypotheses

$$H_0: \sigma_1^2 = \sigma_2^2 \quad \text{vs} \quad H_1: \sigma_1^2 \neq \sigma_2^2$$

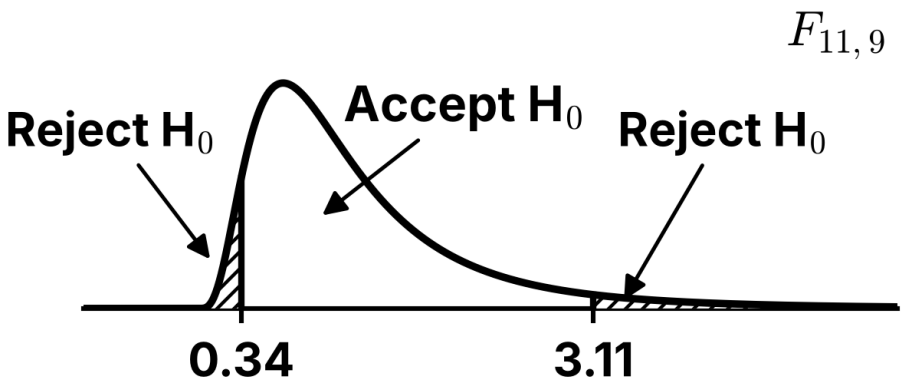
2 Test statistic (The populations are normal)

$$F_{T.S.} = \frac{s_1^2}{s_2^2} = \frac{16}{25} = 0.64 \sim F_{11,\,9}$$

Example 9 Solution (steps 3 and 4)

F

3 Rejection region



Critical values

$$F_{1-\alpha/2,\,11,\,9} = \frac{1}{F_{\alpha/2,\,9,\,11}} = \frac{1}{F_{0.05,\,9,\,11}} = 0.34$$

$$F_{\alpha/2,\,11,\,9} = F_{0.05,\,11,\,9} = 3.11$$

with $\nu_1 = 11$ and $\nu_2 = 9$ degrees of freedom

4 Decision

As $0.34 < F_{T.S.} = 0.64 < 3.11$

we do not reject H_0 and conclude that there is insufficient evidence that the variances differ.