CHAPTER 4

PLANAR GRAPH

When we mention the word "graph" in this chapter, we mean a simple, finite and symmetric graph.

Planar Graphs

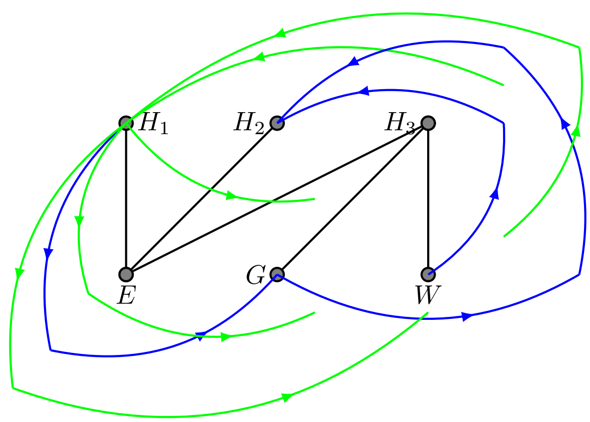
April 11, 2026

1 Introduction

The three Houses and three utilities Problems.

1. Consider three houses (H_1, H_2, H_3) and three utilities outlets (electricity (E), gas (G), water (W)). Is it possible to connect each utility with each of three houses without the lines crossing?

2. **Testing**



For example it appears, in this procedure, to be no way of connecting the third house (H_1) to the water outlet (W)) without crossing some line.
So, either this procedure is incorrect, or there is no way of connecting the houses and the utilities without the lines crossing.

2 Definitions and Examples:

N.B Each graph considered is on at least 1 vertex.

Definition 2.1
A **planar graph** *is a graph that can be drawn in the plane in such a way that no two edges intersect except at a vertex (one end).*

Remark 2.2

The **problem** *mentioned in the introduction is then: Is* $K_{3,3}$ *planar?*

Example 2.3

1. **Example 1:**

   Consider the following graph
   $G = (\{1,2,3,4,5\}, \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\})$.

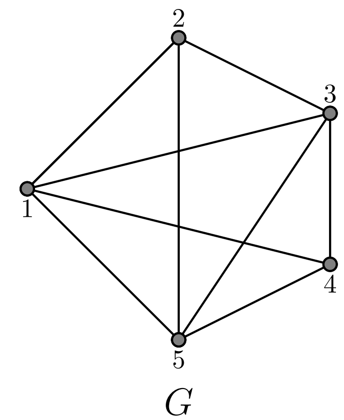


Figure (a)

   Here *G is drawn with intersecting edges.*
   But *G is a planar graph, because G can be drawn as:*

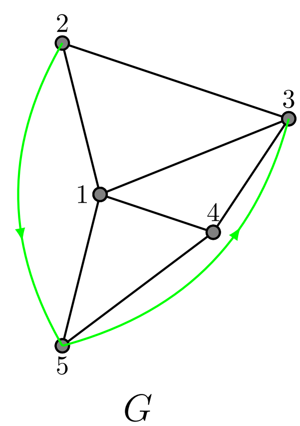


Figure (b)

2. **Example 2:**

   Consider the cube $G = (\{1,2,3,4,5,6,7,8\}, \{\{1,2\},\{1,4\},$
   $\{1,5\},\{2,3\},\{2,6\},\{3,4\},\{3,7\},\{4,8\},\{5,6\},\{5,8\},\{6,7\},\{7,8\}\})$.

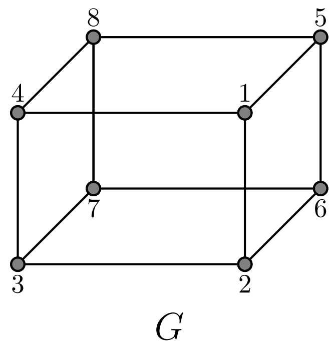


Figure (c)

Here *G is drawn with intersecting edges.*
G can be drawn as:

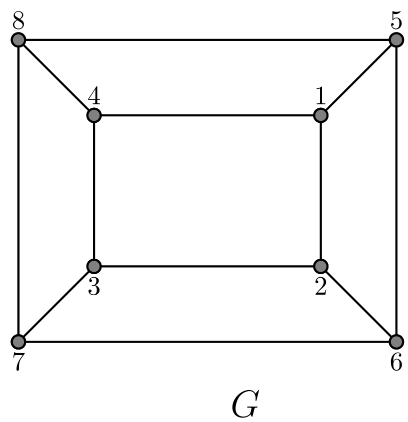
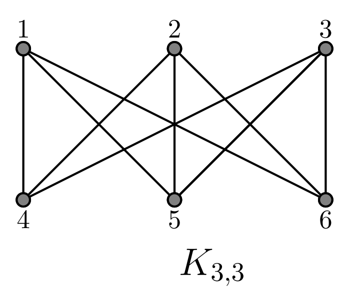


Figure (d)

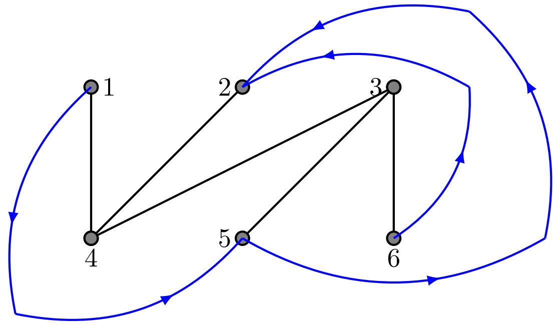
Thus *G is a planar graph.*

3. **Example 3:**

let $K_{3,3}$ be a non planar graph:



For each edge e of $K_{3,3}$, we have: $K_{3,3} \setminus e$ is planar.
For example: $K_{3,3} \setminus \{1, 6\}$ is planar.



$K_{3,3} \setminus \{1,6\}$

Remark 2.4

1. *A planar graph already drawn in the plane so that no two edges intersect (except at a vertex) is referred to as a "plane graph".*
   Thus, the graph of the figure (a) is not a plane graph, but the graph of the figure (b) is a plane graph, the graph of the figure (c) is not a plane graph, but the graph of the figure (d) is a plane graph.

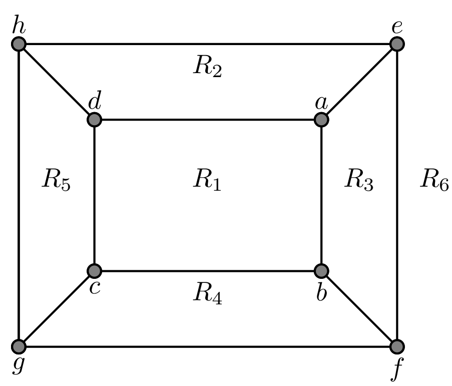
2. *A graph is planar if and only if all its connected components are planar.*

3 Euler's formula

Definition 3.1

Let G be a **plane connected** *graph and consider the parts of the plane remaining after we remove the edges and verticies of G. These "connected pieces" of the plane are called "***region** *of G" (or "***faces** *of G"). The unbounded region (face) is called the "***outer face***". The vertices and edges of G which are incident with a region R make up " The* **boundary** *of R".*

For example, the plane graph (associated with the cube) has 6 regions.



G

Remark 3.2

1. *Every plane connected graph always has exactly one outer face.*

2. *Let G be a connected plane graph and let e be an edge of G.*

(a) *If e is contained in a cycle of G, then e is incident with exactly 2 faces.*

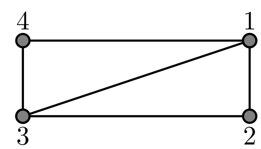
(b) *If e is not contained in any cycle, then it is a bridge (a cut-edge), and e is incident to only one face.*

3. *In a tree, each edge is a bridge, and then a tree has only one face: the outer face.*

Example 3.3

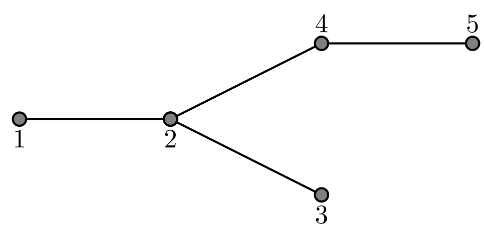
1. *Consider the following three plane graphs:*

   (a) $G_1 = (\{1,2,3,4\}, \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\}\})$



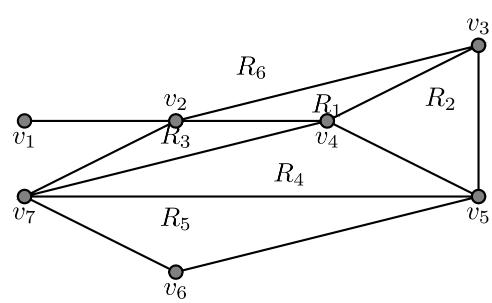
G_1

   (b) $G_2 = (\{1,2,3,4,5\}, \{\{1,2\},\{2,3\},\{3,4\},\{2,5\}\})$



G_2

   (c) $G_3 = (\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\},$
   $\{\{v_1,v_2\},\{v_2,v_3\},\{v_2,v_4\},\{v_2,v_7\},\{v_3,v_4\},\{v_3,v_5\},\{v_4,v_5\},\{v_4,v_7\},\{v_5,v_6\},\{v_5,v_7\},\{v_6,v_7\}\})$



G_3

2. *For $i \in \{1,2,3\}$, denote: $n_i = v(G_i) = |V(G_i)|$ the order of G_i, $e_i = |E(G_i)|$ the size of G_i, and $f_i =$ the number of faces (regions) of G_i.*

3. *So: $f_1 = 3$, $f_2 = 1$, and $f_3 = 6$*

4. *For example, for the face R_6 (outer face of G_3), its boundary consists of the vertices: v_1, v_2, v_3, v_5, v_6, and v_7 and the edges v_1v_2, v_2v_3, v_3v_5, v_5v_6, v_6v_7 and v_2v_7.*

5. *$\forall i \in \{1,2,3\}$, we have $n_i + f_i - e_i = 2$.*

Theorem 3.4 *(***Euler's formula***)*

Let G be a connected plane graph with $n \geq 1$ vertices, e is the number of edges of G and f is the number of faces of G. Then $n + f - e = 2$

Proof.

We use induction on e.

• If $e = 0$. then $n = 1$ and $f = 1$, so, $n + f - e = 1 + 1 - 0 = 2$, and the result is true.

• Let $k \geq 1$ be an integer and assume the result holds for all connected plane graphs with $(k-1)$ edges. Consider a connected plane graph G with: $e = e(G) = k$. Denote: $n = |V(G)|$ and f the number of faces of G.

We will prove that: $n + f - e = 2$.

→ If G is a tree, then ($k = n - 1$ and $f = 1$), so: $n + f - e = n + 1 - (n - 1) = 2$, and the formula follows.

→ Assume that G is not a tree. As G is connected, then G contains cycles (at least one). Let $\alpha = \{u, v\}$ be an edge of G lying on a cycle of G and consider the subgraph: $H = G \setminus \alpha$.

The two faces (of G) incident with the edge α produce one face in subgraph $H = G \setminus \alpha$. Thus, H has n vertices, $(k-1)$ edges and $f - 1$ faces.

So, by our induction hypothesis, we have: $n + (f - 1) - (k - 1) = 2$. Thus, $n + f - k = 2$; which is what we wanted prove.

N. B: From Theorem 3.4, we see that all plane graphs associated with a planar graph G have the same number of faces.

4 Necessary condition and Applications

Theorem 4.1

Let G be a connected planar graph with $n \geq 3$ vertices, e is the number of edges of G. Then $e \leq (3n - 6)$

Proof.

• If $n = 3$, then $e \leq 3$ and the result is true.

• Assume that $n \geq 4$. We draw G as a plane graph and we denote by f the number of faces of G.

→ For each face (region) R of G, let $N(R)$ be the number of edges lying on the boundary of R. Let N be the sum of the number $N(R)$ over all faces of G.

Note that as $n \geq 4$, then $e \geq 3$ (because G is connected). As more, the length of each cycle is at least 3, then for each face R we have: $N(R) \geq 3$. It ensues that: $N \geq 3f$.

On the other hand, as each edge of G is incident with one or two faces, then: $N \leq 2e$ (because N counts every edge of G once or twice). Thus, $3f \leq N \leq 2e$ and then: $3f \leq 2e$ (1).

By Theorem 3.4, we have: $n = 2 - f + e$ (2). Thus, $-f \geq -\frac{2}{3}e$ (by (1)) and then:
$n \geq 2 - \frac{2}{3}e + e$ (by (2)).

Finally, $n \geq 2 + \frac{e}{3}$ and then: $e \leq 3n - 6$.

Remarks 4.2

1. *Intuitively, Theorem 4.1 states that a connected planar graph cannot have too many edges.*
2. *Theorem 4.1 holds for disconnected planar graphs.*

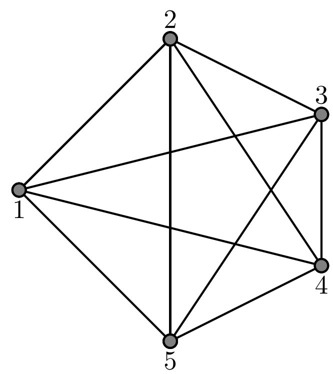
Corollary 4.3

The graph K_5 is not planar.

Proof.

Because in K_5, we have $n = 5$ and $e = 10$, and then: $e = 10 > 3n - 6 = 9$.

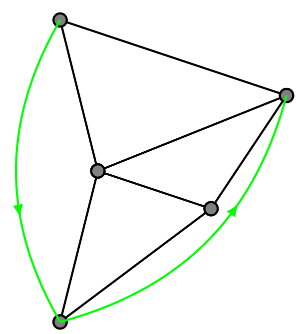
The graph
$K_5 = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\})$.



K_5

Remark 4.4

Here *K_5 is not planar graph.*
But*, for all edge e, $K_5 \setminus e$ is a planar graph, up to isomophy, we can drawn it as:*



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Corollary 4.5

If G is a planar graph, $\delta(G) \leq 5$.

Proof.

If $n = v(G) < 3$, it is trivial. If $n \geq 3$, by Theorem 4.1, $n\delta(G) \leq \sum_{v \in V} d(v) = 2e \leq 6n - 12$. So, $\delta(G) \leq 6 - \frac{12}{n}$. Thus, $\delta(G) \leq 5$.

N. B: For the problem "Three houses and three utilities", we conclude by the following proposition for witch, the proof uses an argument similar to the proof of Theorem 4.1.

Proposition 4.6

The graph $K_{3,3}$ is not planar.

Proof.

Assume, by contradiction, that $K_{3,3}$ is planar. Then we can draw $K_{3,3}$ as a plane graph having f faces.
Consider as the proof of Theorem 4.1, the sum N of the numbers $N(R)$ over all faces of our plane graph $K_{3,3}$.
As $K_{3,3}$, has no triangles (i. e no subgraphs isomorphic to K_3), then the length of each cycle of $K_{3,3}$ is at least 4, and we can see that: $N \geq 4f$. On the other hand: $N \leq 2e = 18$. Thus, $4f \leq 18$ and then $f \leq \frac{9}{2}$ (1).
However, by Theorem 3.4, $n + f - e = n + f - 9 = 2$, so, $6 + f - 9 = 2$. Thus, $f = 5$; which contradicts (1).

Remark 4.7

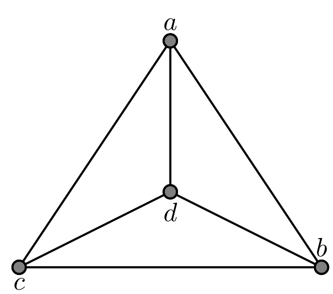
The non planar graphs K_5 and $K_{3,3}$ play a major role in the theory of planar graphs (see the next section).

5 Subdivision and Theorem of Kuratwoski

Definition 5.1

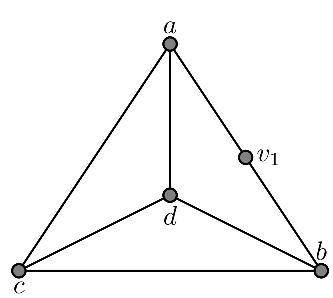
A **subdivision** *of graph G is any graph obtained from G by* **possibly** *inserting vertices (of degree two) into some edges of G.*
So, a subdivision of a graph G is any graph obtained from G by replacing 0, 1 *or more edges by paths of length two or more.*

⋆ **For example**, consider the graph: $G = (\{a, b, c, d\}, \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\})$



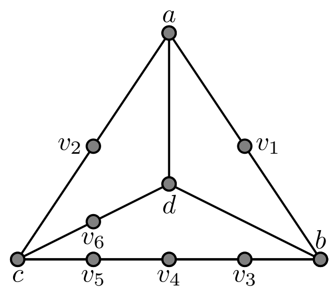
G

Then the graphs G_1 and G_2 are two subdivisions of G, where:
$G_1 = (\{a, b, c, d, v_1\}, \{\{a, v_1\}, \{v_1, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\})$



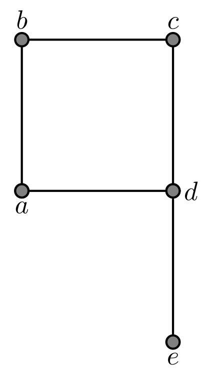
G_1

$G_2 = (\{a, b, c, d, v_1, v_2, v_3, v_4, v_5, v_6\},$
$\{\{a, v_1\}, \{v_1, b\}, \{a, v_2\}, \{v_2, c\}, \{a, d\}, \{b, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, c\}, \{c, v_6\}, \{v_6, d\}\})$



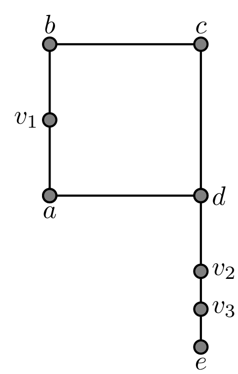
G_2

⋆ **For the graph**: $H = (\{a, b, c, d, e\}, \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{d, e\}\})$



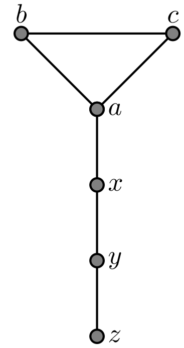
H

We consider
$H_1 = (\{a, b, c, d, v_1, v_2, v_3, e\}, \{\{a, v_1\}, \{v_1, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{d, v_2\}, \{v_2, v_3\}, \{v_3, e\}\})$



H_1

The graphs H_1 is a subdivision of H, while the following graph H_2 is not a subdivision of H.
$H_2 = (\{a, b, c, x, y, z\}, \{\{a, b\}, \{b, c\}, \{c, a\}, \{a, x\}, \{x, y\}, \{y, z\}\})$



H_2

N.B: Clearly the vertices of degree 2 do not affect the planarity of a graph.

Remark 5.2

As K_5 and $K_{3,3}$ are non planar, then if a graph G has a subgraph which is a subdivision of K_5 or $K_{3,3}$, then G is non planar.
Indeed the converse is also true (but proof is too complicated).

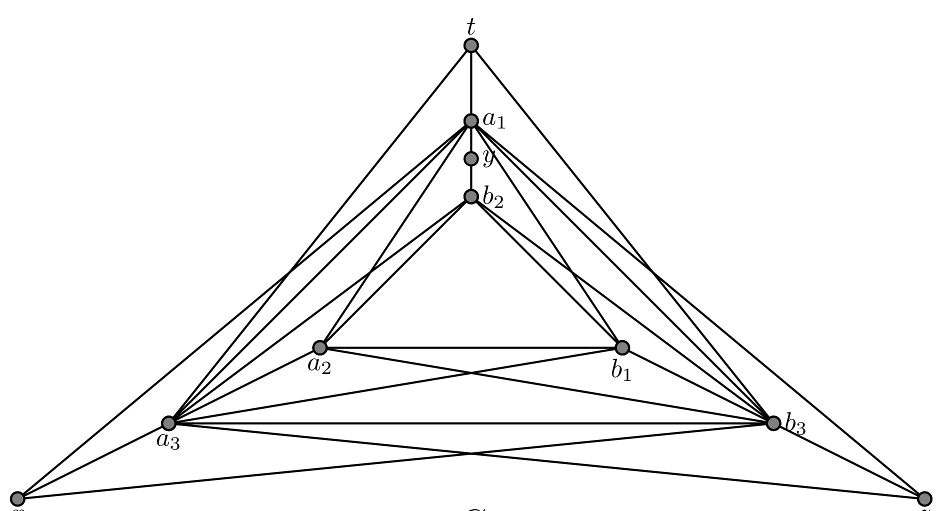
Theorem 5.3 (Kuratowski's Theorem (1930))

A graph is planar if and only if it has no subgraph isomorphic to any subdivision of K_5 or $K_{3,3}$.

Example 5.4

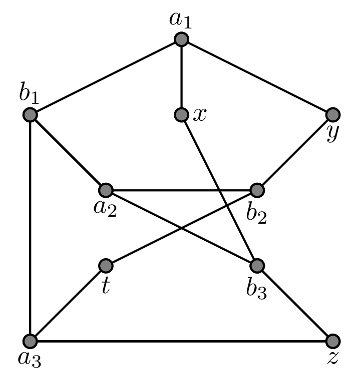
Show that the Petersen graph is non planar.

Indeed: The Petersen graph is isomorphic to:
$G = (\{a_1, z, b_1, 4, b_2, a_2, x, a_3, b_3, y, t\},$
$\{\{a_1, y\}, \{y, b_2\}, \{a_1, a_3\}, \{a_1, b_3\}, \{b_2, z\}, \{b_2, a_3\}, \{b_2, b_3\}, \{b_2, x\}, \{b_2, 4\}, \{b_1, b_2\}, \{b_2, a_2\}, \{t, b_1\},$
$\{t, a_2\}, \{b_1, a_2\}, \{a_3, z\}, \{a_3, b_3\}, \{a_3, x\}, \{a_3, t\}, \{a_3, b_1\}, \{a_3, a_2\}, \{b_3, z\}, \{b_3, x\}, \{b_3, t\}, \{b_3, b_1\}, \{b_3, a_2\}\})$



G

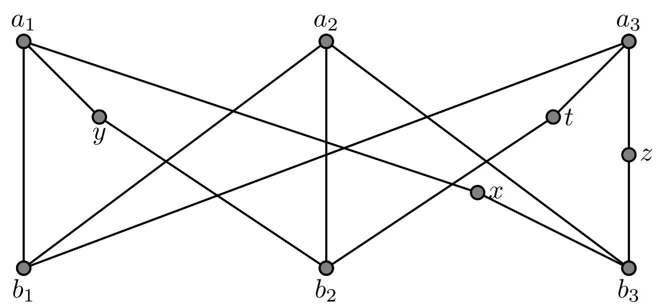
This graph has a subgraph isomorphic to:
$H = (\{a_1, a_2, a_3, b_1, b_2, b_3, x, y, z, t\},$
$\{\{a_1, y\}, \{a_1, x\}, \{a_1, b_1\}, \{y, b_2\}, \{b_2, t\}, \{t, a_3\}, \{a_3, z\}, \{x, b_3\}, \{b_1, a_3\}, \{b_1, a_2\}, \{a_2, b_2\}, \{a_2, b_3\}, \{b_3, z\}\})$



H

But, H is:

$H = (\{a_1, a_2, a_3, b_1, b_2, b_3, x, y, z, t\},$
$\{\{a_1, b_1\}, \{a_1, y\}, \{y, b_2\}, \{a_1, x\}, \{x, b_3\}, \{a_2, b_1\}, \{a_2, b_2\}, \{a_2, b_3\}, \{a_3, b_1\}, \{a_3, t\}, \{t, b_2\}, \{a_3, z\}, \{z, b_3\}\})$



H

which is a subdivision of $K_{3,3}$.

6 EXERCISES OF PLANAR GRAPHS

Exercise 6.1 *Give an example of a graph G satisfying one the following condition.*

1. *G is connected planar graph with $e(G) = 3v(G) - 6$.*
2. *G is connected planar graph with $e(G) < 3v(G) - 6$.*
3. *G is connected non planar graph with $e(G) = 3v(G) - 6$.*
4. *G is 5-regular connected planar graph of order 12.*

Exercise 6.2

1. *Determine all the integers $p \geq 1$ for which K_p is planar.*
2. *Determine all the integers $p, q \geq 1$ for which $K_{p,q}$ is planar.*

Exercise 6.3

1. *Show that if G is a planar graph with $k \geq 1$ connected components and f faces , then $v(G) - e(G) + f = k + 1$.*
2. *Is there any planar graph G with exactly; 36 vertices, 40 edges and 5 faces?*

Exercise 6.4 *Consider a graph G with $v(G) \geq 1$.*

1. *Show that if $v(G) \leq 6$, then G or $\overline{G}$ is planar.*
2. *Show that if $v(G) \geq 11$, then G or $\overline{G}$ is non planar.*
3. *Give an example of a graph H (resp. K) of order 8 such that H and $\overline{H}$ (resp. K and $\overline{K}$) are planar (resp. non planar).*
   N.B: *It is proved that if $v(G) \geq 9$, then G or $\overline{G}$ is not planar (This seems to be hard to proved!!!).*

Exercise 6.5
Let G be a planar graph of order n, and denote by $d(G)$ the average degree of G.

1. *Show that $d(G) < 6$.*
2. *Show that every planar graph G of order at least 4 has at least 4 vertices of degree at most 6.*

Exercise 6.6
Determine the values n for which $\overline{C_n}$ is planar.

Exercise 6.7
Is there any 4-regular planar graph of order 7?

Answer
There is no any 4-regular planar graph of order 7.

Indeed:

First method:
We know this proposition in chapter 2:
Proposition Let $G = (V, E)$ be a connected graph of order $p \geq 2$ such that: $\forall x \in V, d(x) \leq 2$. Then G is a path P_p or a cycle C_p.
In proof of this proposition we remark: "*Let $G = (V, E)$ be a connected graph of order $p \geq 2$ such that: $\forall x \in V, d(x) = 2$. Then G is a cycle C_p.*"
We deduce: "*Let $G = (V, E)$ be a disconnected graph, such that $X_1, ..., X_k$ the k connected components, $|X_i| \geq 2$, and $\forall x \in V, d(x) = 2$. Then G is the union of k cycles $C_{p_i}, i \in \{1, ..., k\}$.*

In this exercise, we assume that $G = (V, E)$ is a 4-regular graph of order 7 and $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, then $\overline{G} = (V, \overline{E})$ is a $(7 - 1 - 4) = 2$-regular graph of order 7.

Case 1: $\overline{G} = (V, \overline{E})$ **is connected**, then $\overline{G} = C_7$, it follows $G = \overline{C_7}$, is contains a subdivision of $K_{3,3}$, hence G is not planar graph.

Case 2: $\overline{G} = (V, \overline{E})$ **is disconnected**, since $\overline{G}$ is a 2- regular graph, then $\overline{G}$ is the union of cycles with 7 vertices, since a cycle contains at least 3 vertices, it follows $\overline{G}$ is the union of 2 cycles C_3 and C_4, such that of one is 3-cycle and the other is a 4-cycle, suppose that $C_3 = \overline{G}[\{v_1, v_2, v_3\}]$ and $C_4 = \overline{G}[\{v_4, v_5, v_6, v_7\}]$. Then $G[\{v_1, v_2, v_3, v_4, v_6, v_7\}] \setminus \{v_4, v_7\}$ and $K_{3,3}$ are isomorphic, hence G is not planar graph.

Second method:

Assume that $G = (V, E)$ is a 4-regular graph of order 7 and $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, without loss of generality, we suppose that: $N_G(v_1) = \{v_2, v_3, v_4, v_5\}$. We have the vertices v_6 and v_7 non adjacent with the vertex v_1.
Case 1: Vertices v_6 and v_7 are not adjacent, since G is a 4-regular graph, then $N_G(v_6) = N_G(v_7) = \{v_2, v_3, v_4, v_5\}$, hence the induced subgraph $G[\{v_1, v_2, v_3, v_4, v_6, v_7\}]$ of G and $K_{3,3}$ are isomorphic, therefore G is not planar (consider $\{x\} = \{v_2, v_3, v_4, v_5\} \setminus A$, since G is 4-regular graph, then $\{x, y\} \in E, \forall y \in A$).
Case 2: Vertices v_6 and v_7 are adjacent, since G is a 4-regular graph, then $|N_G(v_6) \cap \{v_2, v_3, v_4, v_5\}| = 3$ and $|N_G(v_7) \cap \{v_2, v_3, v_4, v_5\}| = 3$, we have two subcases:
• **Subcase 1:** $|N_G(v_6) \cap N_G(v_7)| = 3$, and suppose that $N_G(v_6) \cap N_G(v_7) = A \subset \{v_2, v_3, v_4, v_5\}$, hence the subgraph $G[\{v_1, v_6, v_7\} \cup A] \setminus \{v_6, v_7\}$ of G and $K_{3,3}$ are isomorphic, therefore G is not planar.
• **Subcase 2:** $|N_G(v_6) \cap N_G(v_7)| = 2$, and suppose that $N_G(v_6) \cap N_G(v_7) = B \subset \{v_2, v_3, v_4, v_5\}$, we suppose that: $\{v_2, v_3, v_4, v_5\} \setminus B = \{\alpha, \beta\}$, $\alpha \in N_G(v_6)$, $\beta \in N_G(v_7)$, the two vertices of $B = \{a, b\}$ are not adjacent (because if not and we have G is a 4-regular graph, then $N_G(a) = \{b, v_1, v_6, v_7\}$ and $N_G(b) = \{a, v_1, v_6, v_7\}$, it follows $d(\alpha) < 4$ contradicts with G is a 4-regular graph). Therefore, $N_G(v_6) = \{\alpha, v_7, a, b\}$ and $N_G(v_7) = \{\beta, v_6, a, b\}$, hence the two vertices of $\{\alpha, \beta\}$ are adjacent, therefore, the graph G contains a subdivision of $K_{3,3}$, then G is not planar.

Remark: The relations between the first method and the second method. We consider the complement $\overline{G}$ of G in the second method:
1- We find in case 1, the $\overline{G}$ of G is the union of 2 cycles C_3 and C_4.
2- We find in case 2 and subcase 1 the $\overline{G}$ of G is the union of 2 cycles C_3 and C_4.
3- We find in case 2 and subcase 2 the $\overline{G}$ of G is the cycle C_7.