Chapter 4: Power Series (Taylor, Differentiation, Integration)

Power Series

Definition and convergence set

A **power series** centered at $x_0 \in \mathbb{R}$ is

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

For each fixed x, this is a numerical series.

The **domain of convergence** is

$$E := \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} c_n (x - x_0)^n \text{ converges} \right\},$$

and on E it defines a function $f(x) = \sum c_n (x - x_0)^n$.

Cauchy–Hadamard Radius of Convergence

Theorem

Let

$$d := \limsup_{n\to\infty} |c_n|^{1/n} \in [0, \infty], \qquad R := \frac{1}{d} \quad (R = 0 \text{ if } d = \infty, \;\; R = \infty \text{ if } d = 0).$$

Then:

1. If $|x - x_0| < R$, the series converges absolutely.
2. If $|x - x_0| > R$, the series diverges.
3. If $0 < R < \infty$ and $|x - x_0| = R$, boundary points must be checked separately.

Proof idea (root test)

Proof (idea)

Apply the root test to $a_n(x) = c_n(x - x_0)^n$:

$$\limsup_{n\to\infty} |a_n(x)|^{1/n} = |x - x_0| \limsup_{n\to\infty} |c_n|^{1/n} = |x - x_0|\, d.$$

Root test yields absolute convergence if $|x - x_0|d < 1$ and divergence if $|x - x_0|d > 1$. The boundary $|x - x_0|d = 1$ corresponds to $|x - x_0| = R$.

Uniform convergence on compact subsets

Theorem

Let R be the radius of convergence of $\sum c_n(x-x_0)^n$. For every ρ with $0<\rho<R$, the series converges uniformly on the closed interval

$$[x_0-\rho,\ x_0+\rho].$$

Proof idea (*M*-test)

Proof (idea)

Fix $0 < \rho < R$ and x with $|x - x_0| \leq \rho$. Then

$$|c_n(x - x_0)^n| \leq |c_n|\rho^n.$$

Since $\sum |c_n|\rho^n$ converges (because $\rho < R$), the Weierstrass *M*-test gives uniform convergence.

Termwise differentiation (power series)

Theorem

Let $R > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} c_n(x-x_0)^n$. Define on $(x_0 - R, x_0 + R)$

$$f(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n.$$

Then f is differentiable on $(x_0 - R, x_0 + R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n(x-x_0)^{n-1}.$$

Moreover, the derived series has the **same radius** R.

Proof idea (uniform convergence on $[x_0 - \rho, x_0 + \rho]$)

Proof (idea)

Fix $0 < \rho < R$. On $[x_0 - \rho, x_0 + \rho]$ both series

$$\sum c_n (x - x_0)^n \quad \text{and} \quad \sum n c_n (x - x_0)^{n-1}$$

converge uniformly (use M-test with bounds $|c_n|\rho^n$ and $n|c_n|\rho^{n-1}$). Uniform control of the difference quotients implies we may differentiate term-by-term on $(x_0 - \rho, x_0 + \rho)$. Since $\rho < R$ is arbitrary, the identity holds on $(x_0 - R, x_0 + R)$.

Termwise integration (power series)

Theorem

Let $R > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} c_n(x-x_0)^n$. For any x with $|x-x_0| < R$,

$$\int_{x_0}^{x} \left(\sum_{n=0}^{\infty} c_n(t-x_0)^n \right) dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-x_0)^{n+1}.$$

The integrated series also has radius R.

Proof idea (uniform convergence + termwise integration)

Proof (idea)

Fix $0 < \rho < R$. On $[x_0 - \rho, x_0 + \rho]$, $\sum c_n(t - x_0)^n$ converges uniformly. Hence we may integrate term-by-term on this interval:

$$\int \sum c_n(t - x_0)^n \, dt = \sum \int c_n(t - x_0)^n \, dt.$$

Since $\rho < R$ is arbitrary, the identity holds for all $|x - x_0| < R$.

Taylor coefficients

Derivatives determine the coefficients

Let

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad (|x - x_0| < R).$$

By repeated termwise differentiation,

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)c_k(x - x_0)^{k-n}.$$

Evaluating at $x = x_0$ gives the **Taylor formula for coefficients**:

$$f^{(n)}(x_0) = n!\, c_n \qquad \Longrightarrow \qquad c_n = \frac{f^{(n)}(x_0)}{n!}.$$

Taylor series of a power series (exact equality)

Theorem

If $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ on $|x - x_0| < R$, then for all $|x - x_0| < R$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

So every power series equals its Taylor series (inside its radius).

Standard power series (center 0)

Example

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad (R = \infty)$$

Summary

What you must remember

- Radius R from Cauchy–Hadamard: absolute convergence for $|x - x_0| < R$, divergence for $|x - x_0| > R$.
- Uniform convergence on every compact sub-interval $[x_0 - \rho, x_0 + \rho]$ with $\rho < R$.
- Inside $|x - x_0| < R$: **termwise differentiation and termwise integration** are valid.
- Coefficients satisfy $c_n = \dfrac{f^{(n)}(x_0)}{n!}$, so the power series equals its Taylor series.