Chapter 4

Randomized Blocks
Latin Square

4.1 The Randomized Complete Block Design

Randomized complete block design (RCBD)

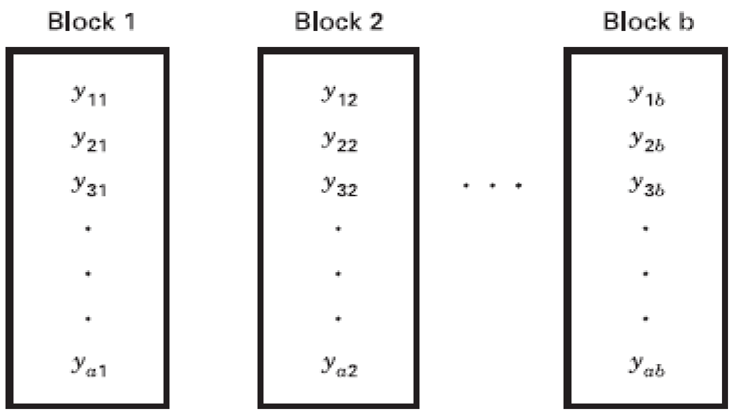
We define a **nuisance factor** as a design factor that probably has an effect on the response, but we are not interested in that effect. When the nuisance source of variability is known and controllable, a design technique called **blocking** can be used to systematically eliminate its effect on the statistical comparisons among treatments. **Blocking** is an extremely important design technique used extensively in industrial experimentation and is the subject of this chapter. We would like to make the **experimental error as small as possible**; that is, we would like to remove the variability between blocks from the experimental error. The design, shown in Table 4.1, is called a **randomized complete block design** (**RCBD**).

■ TABLE 4.1
Randomized Complete Block Design for the Hardness Testing Experiment

Test Coupon (Block)			
1	2	3	4
Tip 3	Tip 3	Tip 2	Tip 1
Tip 1	Tip 4	Tip 1	Tip 4
Tip 4	Tip 2	Tip 3	Tip 2
Tip 2	Tip 1	Tip 4	Tip 3

4.1.1 Statistical Analysis of the RCBD

Suppose we have, in general, *a* treatments that are to be compared and *b* blocks. The randomized complete block design is shown in the following Figure



■ **FIGURE 4.1** The randomized complete block design

There is one observation per treatment in each block. The statistical model for the **RCBD** can be written in several ways.

Fixed Effects model: $$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}, \quad \begin{cases} i=1,2,\ldots,a \\ j=1,2,\ldots,b \end{cases} \qquad \textbf{(4.1)}$$

where μ is an overall mean, τ_i is the effect of the i *th* treatment, β_j is the effect of the j *th* block, and ε_{ij} is the usual $NID\ (0, \sigma^2)$ random error term. We will initially consider treatments and blocks to be **fixed factors**.

Model for the $\boldsymbol{RCBD}$ is an overspecified model. The treatment and block effects denoted as deviations from the overall mean so that: $\sum_{i=1}^{a} \tau_i = 0, \ \sum_{j=1}^{b} \beta_j = 0$

$\boldsymbol{Mean\ model}$: $$y_{ij} = \mu_{ij} + \varepsilon_{ij}, \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \end{cases}$$

where $\mu_{ij} = \mu + \tau_i + \beta_j$

In the $\boldsymbol{RCBD}$, we are interested in testing the equality of the **treatment means**. Thus, the hypotheses of interest are

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_a \,, \qquad H_1: \mu_i \neq \mu_j \text{ for at least one pair } (i, j)$$

an equivalent way to write the above hypotheses is in terms of the **treatment effects**, say

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0 \,, \qquad H_1: \tau_i \neq 0 \text{ for at least one of } i$$

Form of data in *RCBD*

Suppose we have a treatments, b blocks. The data would appear as in following table

Data in case of ***RCBD***

Treatment	Blocks				Total $y_{i\cdot}$	Mean $\bar{y}_{i\cdot}$
	1	2	...	b		
1	y_{11}	y_{12}	$\cdots$	y_{1b}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	$\cdots$	y_{2b}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{ab}	$y_{a\cdot}$	$\bar{y}_{a\cdot}$
Total: $y_{\cdot j}$	$y_{\cdot 1}$	$y_{\cdot 2}$	...	$y_{\cdot b}$	$y_{\cdot\cdot}$	$\bar{y}_{\cdot\cdot}$
Mean: $\bar{y}_{\cdot j}$	$\bar{y}_{\cdot 1}$	$\bar{y}_{\cdot 2}$	...	$\bar{y}_{\cdot b}$	$\bar{y}_{\cdot\cdot}$	

▪ Parameters Estimation of Fixed Effect Model

Table 3.2: Estimation of Model Parameters

Parameter		Estimat
Treatment Mean	μ_{ij}	$\hat{\mu}_{ij} = \bar{y}_{i.} + \bar{y}_{.j} - \bar{y}_{..}$
Overall mean	μ	$\bar{y}_{..} = \frac{y_{..}}{N} = \frac{y_{..}}{ab}$
Treatment effect	τ_i	$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$
block effect	β_j	$\hat{\beta}_j = \bar{y}_{.j} - \bar{y}_{..}$
Variance of errors	σ^2	$\hat{\sigma}^2 = MS_E = \frac{SS_E}{N-a}$

We may express the total corrected sum of squares SS_T as

$SS_T =$	$SS_{Treatment} +$	$SS_{block} +$	SS_E	
$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 =$	$b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 +$	$a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2 +$	$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2$	(4.7)
$\sum_{i=1}^{a}\sum_{j=1}^{b}y_{ij}^2-\frac{y_{..}^2}{N}$	$\frac{1}{b}\sum_{i=1}^{a}y_{i.}^2-\frac{y_{..}^2}{N}$	$\frac{1}{a}\sum_{j=1}^{b}y_{.j}^2-\frac{y_{..}^2}{N}$	$SS_T-SS_{Treatment}-SS_{block}$	**4.9 - 4.12**
Total Sum of Squares	**Treatment Sum of Squares**	**Block Sum of Squares**	**Errors Sum of Squares**	
$N-1$	$a-1$	$b-1$	$(a-1)(b-1)$	$\leftarrow df$

○ ANOVA table

The Analysis of Variance Table for the ***RCBD***, Fixed Effects Model

Source of Variations	Sum of Squares	Degrees of Freedom	Mean Squares	F_0
Treatments	$SS_{Treatments}$	$a-1$	$MS_{Treatment}$	$F_{Tr} = \frac{MS_{Treatments}}{MS_E}$
Blocks	SS_{block}	$b-1$	SS_{block}	
Errors	SS_E	$(a-1)(b-1)$	MS_E	
Total	SS_T	$(N-1)$		

Ex1: A chemist wishes to test the effect of four chemical agents on the strength of a particular type of cloth. Because there might be variability from one bolt to another, the chemist decides to use a randomized block design, with the bolts of cloth considered as blocks. She selects five bolts and applies all four chemicals in random order to each bolt. The resulting tensile strengths follow.

	Bolt (Blocks)					
Chemical	**1**	**2**	**3**	**4**	**5**	$\bar{y}_{i\cdot}$
1	73	67	73	70	66	**349**
2	73	67	75	72	70	**357**
3	75	68	78	73	68	**362**
4	73	71	75	75	69	**363**
	$\bar{y}_{\cdot j}$	**294**	**273**	**301**	**290**	**1431**

Show whether the treatments have an significant effect on tensile strength. Use $\alpha = 0.05$.

- ## Compute Sum of Squares

$$Q_0 = CF = \frac{y_{..}^2}{N} = \frac{1431^2}{20} = 102388.05$$

$$Q_{To} = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 = 73^2 + 67^2 + \cdots + 69^2 = 102597$$

$$Q_{Tr} = \frac{1}{b}\sum_{i=1}^{4} y_{i\cdot}^2 = \frac{1}{5}(349^2 + 357^2 + 362^2 + 363^2) = \frac{512063}{5} = 102412.6$$

$$Q_{Bl} = \frac{1}{a}\sum_{i=1}^{4} y_{\cdot j}^2 = \frac{1}{4}(294^2 + 273^2 + 301^2 + 290^2 + 373^2) = \frac{410195}{4} = 102548.75$$

==

$$SS_T = Q_{To} - Q_0 = 102597 - 102388.05 = 208.95$$

$$SS_{Treatment} = Q_{Tr} - Q_0 = 102412.6 - 102388.05 = 24.55$$

$$SS_{block} = Q_{Bl} - Q_0 = 102548.75 - 102388.05 = 160.7$$

$$SS_E = SS_T - SS_{Treatments} - SS_{block} = 208.95 - 24.55 - 160.7 = 23.7$$

ANOVA table

Source of Variation	***SS***	***df***	***MS***	***F***	***P-value***	***F crit***
Chim. (Treatment)	24.55	3	8.183	4.143	0.031	3.490
Bolt (Block)	160.7	4	40.175	20.342	2.81E-05	3.259
Error	23.7	12	1.975			
Total	208.95	19				

- **Steps of hypothesis test**

$H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4$ against $H_1: \tau_i \neq 0$ for at least one of i

$F_{Tr} = 4.143$

$F_{\alpha, a-1, N-a} = F_{0.05,3,12} = 3.49$

Because $F_{Tr} = 4.143 > F_{0.05,3,12} = 3.49$, we reject H_0 and conclude that the treatment have significant effect at significant level 0.05

==

Ex2: By using data in Ex1, Perform pairwise comparisons between processors using the least significant difference (*LSD*). Use $\alpha = 0.05$

- Compute *LSD*

$$LSD = t_{\alpha/2,\,(a-1)(b-1)}\sqrt{2MS_E/b} = t_{0.025,\,12}\sqrt{2 \times 1.975/5} = 2.179 \times 0.889 = \mathbf{1.937}$$

- **Table of comparisons**

$\bar{y}_{i\cdot}$	$\bar{y}_{i`\cdot}$	$\lvert\bar{y}_{i\cdot} - \bar{y}_{j\cdot}\rvert$	LSD	H_0	Decision
69.8	71.4	1.6	1.937	$H_0: \mu_1 = \mu_2$	H_0 is not rejected
69.8	72.4	2.6	1.937	$H_0: \mu_1 = \mu_3$	H_0 is rejected
69.8	72.6	2.8	1.937	$H_0: \mu_1 = \mu_4$	H_0 is rejected
71.4	72.4	1.0	1.937	$H_0: \mu_2 = \mu_3$	H_0 is not rejected
71.4	72.6	1.2	1.937	$H_0: \mu_2 = \mu_4$	H_0 is not rejected
72.4	72.6	0.2	1.937	$H_0: \mu_3 = \mu_4$	H_0 is not rejected

Estimating Missing Values (page 154)

The missing value x as shown in following table is estimated as

Treatment	Blocks				Total
	1	2	j	b	$y_{i\cdot}$
1	y_{11}	y_{12}	y_{1j}	y_{1b}	$\mathbf{y_{1\cdot}}$
…	…	…	…	…	…
i	y_{i1}	y_{i2}	x	y_{ib}	$y'_{i\cdot}$
…	…	…	…	…	…
a	y_{a1}	y_{a2}	y_{aj}	y_{ab}	$\boldsymbol{y_{a\cdot}}$
Total: $y_{\cdot j}$	$y_{\cdot 1}$	$y_{\cdot 2}$	$y'_{\cdot j}$	$y_{\cdot b}$	$y'_{\cdot\cdot}$

$$x = \frac{a y'_{i\cdot} + b y'_{\cdot j} - y'_{\cdot\cdot}}{(a-1)(b-1)}$$

	Bolt (Blocks)					$y_{i\cdot}$
1	73	67	73	70	66	349
2	73	67	75	72	70	357
3	75	68	x	73	68	284
4	73	71	75	75	69	363
$y_{\cdot j}$	294	273	223	290	273	1353

$$x = y_{33} = \frac{a y'_{3\cdot} + b y'_{\cdot 3} - y'_{\cdot\cdot}}{(a-1)(b-1)}$$

$$x = \frac{4\times 284 + 5\times 223 - 1353}{(3)(4)}$$

=74.8

Using R Program Software with CRBD

Ex3: Use R program software for obtaining the results of $CRBD$, use data in Ex1:

The program:

```
Y = c(73,67,73,70,66,73,67,75,72,70,75,68,78,73,68,73,71,75,75,69)

Chem = factor(rep(c("Chem1", "Chem2", "Chem3","Chem4"), each = 5));    # Tratments

Bolt = factor(rep(c("Bol1", "Bol2", "Bol3", "Bol4","Bol5"),4));    # Blocks

 data1 <- data.frame(Y,Chem,Bolt)

Model = aov(Y~Chem+Bolt, data=data1); summary(Model)


#Pairwise comparison to treatments (Chem) using LSD

library(agricolae)

LSD = LSD.test(Model, "Chem"); print(LSD)
```

```
> summary(Model)
            Df Sum Sq Mean Sq F value   Pr(>F)
Chem         3  24.55    8.18   4.143   0.0313 *
Bolt         4 160.70   40.17  20.342 2.81e-05 ***
Residuals   12  23.70    1.97
> print(LSD)
$statistics
  MSerror Df  Mean       CV  t.value      LSD
    1.975 12 71.55 1.964147 2.178813 1.936571

Fisher-LSD      none   Chem   4  0.05
$means
         Y      std r        se      LCL      UCL Min Max Q25 Q50 Q75
Chem1 69.8 3.271085 5 0.6284903 68.43064 71.16936  66  73  67  70  73
Chem2 71.4 3.049590 5 0.6284903 70.03064 72.76936  67  75  70  72  73
Chem3 72.4 4.393177 5 0.6284903 71.03064 73.76936  68  78  68  73  75
Chem4 72.6 2.607681 5 0.6284903 71.23064 73.96936  69  75  71  73  75
Y groups
Chem4 72.6      a
Chem3 72.4      a
Chem2 71.4     ab
Chem1 69.8      b
```