Eulerian Graphs-Hamiltonian Graphs

April 11, 2026

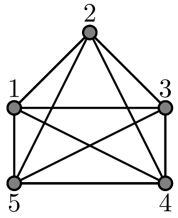
1 Eulerian Graphs

1.1 Definitions:

1. A *tour* of a connected graph G is a closed walk that traverses each edge of G at least once.
2. An *Euler tour* is a tour traversing each edge of G exactly once (i.e a closed Euler trail).
3. A graph is *eulerian* if it admits an Euler tour.

1.2 Examples:

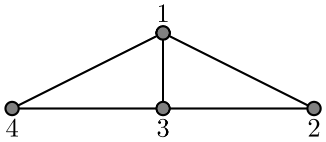
1. $G_1 \simeq K_5$ is an eulerian graph
   $G_1 = (\{1,2,3,4,5\}, \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\})$

   $G_1 \simeq K_5$

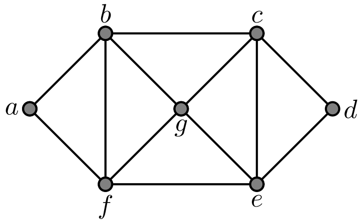
   Indeed: $C = (1,2,3,1,5,4,3,5,2,4,1)$ is an Euler tour.

2. $G_2 = (\{1,2,3,4\}, \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\}\})$ is non eulerian graph (You can verify).

   G_2

3. $G_3 = (\{a,b,c,d,e,f,g\},$
   $\{\{a,b\},\{a,f\},\{b,c\},\{b,g\},\{b,f\},\{c,d\},\{c,e\},\{c,g\},\{d,e\},\{e,g\},\{e,f\},\{f,g\}\})$ is an eulerian graph.



G_3

Consider: $C = (a, b, c, d, e, c, g, f, e, g, b, f, a)$ is an Euler tour.

Remark 1.1

Consider an eulerian graph G *and an Euler tour* $W = (u = \alpha_1, \alpha_2, ..., \alpha_p, u)$. *Each time a vertex* v *occurs as an internal vertex of* W, *2 edges incident with* v *are accounted for. But* W *traverses each edge exactly once. Then* $\forall u \neq v$, $d_G(v)$ *is even. Also,* $d_G(u)$ *is even, because* W *starts and ends at* u. *Thus,* G *is even.*

Theorem 1.2 *(***Euler 1736***).*

A connected graph is eulerian if and only if it is even.

Proof.

1. " $\Rightarrow$ " If a connected graph is eulerian, then it is even by Remark.
2. " $\Leftarrow$ " Consider the converse statement.
   Assume that $G = (V, E)$ is a connected and even graph.
   → Select a **vertex** u of G and begin a **Trail** P at u. We continue this trail as long as possible (until we reach a vertex w such that the only edges incident with w already belong to P, hence, P cannot be continued, and we must stop).
   Fact 1: $w = u$ (i.e P is a closed trail).

   Indeed:

   If $w \neq u$, clearly an odd number of edges incident with w appears on P. As $d_G(w)$ is even, there is at least one edge incident with w that does not belong to P. Thus, P can be continued; contradiction.
   Fact 2: If $E(P) \subsetneq E(G)$, then there is one vertex (at least one) v on P that is incident with an edge $e \in E(G) \setminus E(P)$ (at least one edge e).

   Indeed:

   - If P contains all vertices of G, each $e \in E(G) \setminus E(P)$ satisfies the property.
   - If not, let X be the set of vertices of P. As $X \in \mathcal{P}(V) \setminus \{\emptyset, V\}$, then the edge cut of G, $\partial(X)$, associated with X is nonempty. Let $e = \{x, y\} \in \partial(X)$ where $x \in X$ (and $y \notin X$). So the vertex x and the edge e satisfy the property.

   N.B

   - If $E(P) = E(G)$, then P is an Euler tour.
   - Assume that, $E(P) \subsetneq E(G)$.

→ Now, remove the edges of P and consider the subgraphs $H = G \setminus E(P) = (V, E(G) \setminus E(P))$. Clearly, every vertex in H has even degree. Let $H_1 = H[X_1]$ where X_1 is the connected component of H containing the vertex v (the vertex v is choose in Fact 2). We begin a trail P_1 in H_1 at v and continue this trail as long as possible. As before, P_1 is a closed trail.

Consider the closed trail $P = (\alpha_1 = u, \alpha_2, ..., \alpha_{\mathbf{i}} = \mathbf{v}, \alpha_{i+1}, ..., \alpha_p, u)$.

By inserting the closed trail P_1 at a place "$\alpha_i = v$", we obtain a closed trail C_1 of G, beginning and ending at u, which has more edges than P.

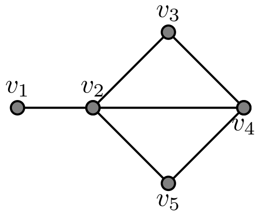
- If C_1 contains all the edges of G, then C_1 is an Euler tour, and G is an eulerian graph.
- If C_1 does not contain all the edges of G, then we may continue the above procedure until we finally obtain an Euler tour.

Definition 1.3 *(***Traversable graph***).*

If a connected graph G has a trail, which is not closed, and containing all edges of G, then G is called: "a traversable graph" and the trail is called: "an Euler trail".

Example 1.4

$G = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}\})$.



G

The graph G is traversable and $P = (v_1, v_2, v_4, v_3, v_2, v_5, v_4)$ *is an euler trail.*

Theorem 1.5

A connected graph G is traversable if and only if G has exactly two odd vertices. Furthermore, any euler trail of G begins at one of the odd vertices and ends at the other odd vertex.

Proof.

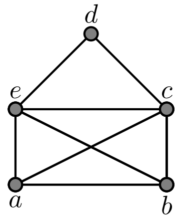
- " ⇒ " It is clear.
- " ⇐ " Consider a connected graph G with exactly two odd vertices: a and b. Consider the graph $\widetilde{G} = (V(G) \cup \{z\}, E(G) \cup \{\{a, z\}, \{b, z\}\})$ where $z \notin V(G)$.
  By Theorem 1.2, $\widetilde{G}$ is an eulerian graph and then there is a closed trail $C = (\alpha_1 = a, \alpha_2 = z, \alpha_3, ..., \alpha_p = a)$. So $\alpha_3 = b$ and $P = (\alpha_3 = b, \alpha_4, ..., \alpha_p = a)$ is an euler trail of G, hence, G is a connected graph and traversable.

Remark 1.6

An interesting property of eulerian and traversable graphs is that: Once the vertices have been drawn, we can draw the edges in one "continuous" motion (i.e the edges can be drawn without lifting the pencil from the paper). Note that the condition is: the number of odd vertices is zero or two.

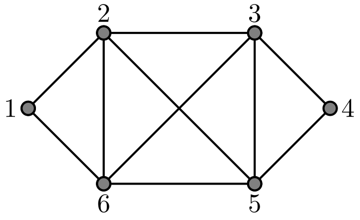
Example 1.7

1. *"Envelope": $G = (\{a,b,c,d,e\}, \{\{a,b\},\{a,c\},\{a,e\},\{b,c\},\{b,e\},\{c,d\},\{c,e\},\{d,e\}\})$ We can draw this envelope without lifting the pencil from the paper, but for this, we must begin by one of $\{a,b\}$ and end at the other of $\{a,b\}$.*
   (For example: $P = (a,c,e,b,a,e,d,c,b)$).



G

2. $G = (\{1,2,3,4,5,6\}, \{\{1,2\},\{1,6\},\{2,3\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{5,6\}\})$
   G is an eulerian graph because it connected and even.



G

Find an Euler tour:

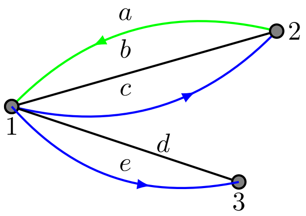
- *We begin at $u = 1$. we obtain the closed trail $C_1 = (1,2,6,1)$.*
- *The edge $\{6,5\}$ is not in C_1 (and 6 is a vertex of C_1).*
- *In $G \setminus E(C_1)$, we begin at 6, and we obtain $C_2 = (6,5,4,3,2,5,3,6)$.*
- *Insertion of C_2 in C_1, we obtain $C_3 = (1,2,6,5,4,3,2,5,3,6,1)$ is an Euler tour of G.*

Remarks 1.8 *(***The seven bridges of Königsberg***)*

1. *In 1736, Euler showed that it was impossible to cross each of the seven bridges of Königsberg once and only once during a walk through the town.*
   Here is a plan of Königsberg and the river Pregel:

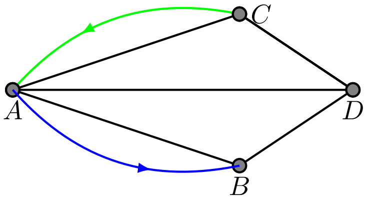
2. *For this we need the notion "multigraph".*

   - *In a multigraph, we can repeat some edges.*
     Example:
     $M = (\{1,2,3\}, \{a = \{1,2\}, b = \{1,2\}, c = \{1,2\}, d = \{1,3\}, e = \{1,3\}\})$.



   - *We can define the notions: walk, trail, connected multigraph.*

3. *For the problem of the seven bridges of Königsberg, consider the multigraph:*
   ($G = (\{A,B,C,D\}, \{\{A,B\}, \{A,B\}, \{A,C\}, \{A,C\}, \{A,D\}, \{C,D\}, \{B,D\}\})$*)*



   The multigraph G *has* 4 *vertices of odd degree. So* G *is neither eulerian graph, nor traversable graph.*

2 Hamiltonian Graphs

2.1 Definitions:

1. A path (resp. A cycle) of graph G is called a *Hamilton path* (resp. *Hamilton cycle*) if it contains every vertex of G.

2. A graph is *traceable* (resp. *hamiltonian*) if it contains a Hamilton path (resp. Hamilton cycle).

Remark 2.1

1. *If a graph* $G = (V, E)$ *is hamiltonian, then:*

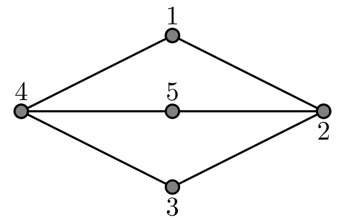
   (a) $|V| \geq 3$.

   (b) $\forall x \in V,\ d_G(x) \geq 2$.

   (c) G *is connected.*

   (d) G *has no cut-vertex.*

2. Consider the following graph: $G = (\{1,2,3,4,5\}, \{\{1,2\},\{1,4\},\{2,3\},\{2,5\},\{3,4\},\{4,5\}\}$.

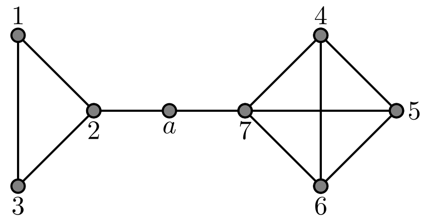


G

This graph G *satisfies the* 4 *conditions on the first remark. However this graph is nonhamiltonian.*

Example 2.2

$G = (\{1,2,3,4,5,6,7,a\}, \{\{1,2\},\{2,3\},\{3,1\},\{2,a\},\{a,7\},\{4,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}\})$, *is non hamiltonian graph because the vertex a is a cut-vertex.*



G

2.2 Properties of hamiltonian graph:

As a **necessary condition** we have the following theorem

Theorem 2.3

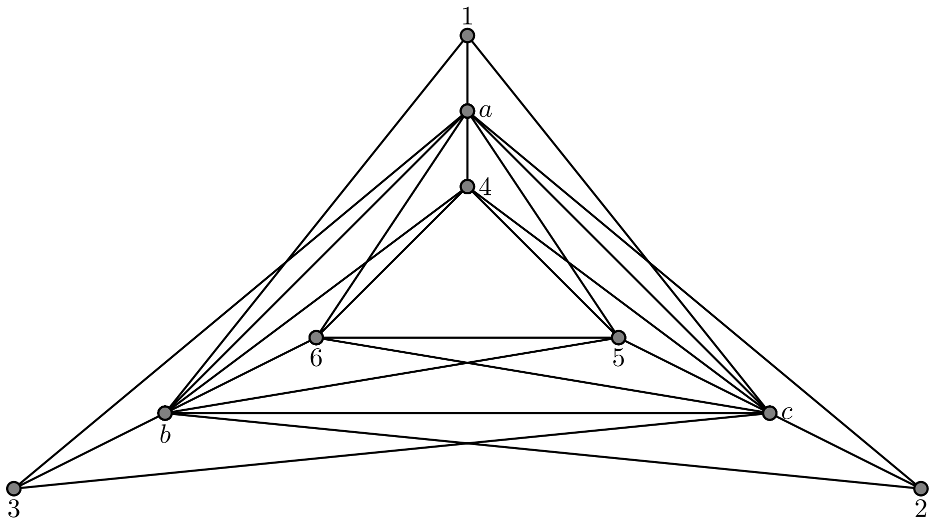
Given a hamiltonian graph $G = (V,E)$*, then for every nonempty proper subset* S *of* V*,* $c(G-S) \leq |S|$*. Moreover, if equality holds, then each of the* $|S|$ *connected components of* $G-S$ *is traceable, and every Hamilton cycle of* G *includes a Hamilton path in each of these components.*

Proof.

- Let C be a Hamilton cycle. Clearly $C-S$ has at most $|S|$ components and, since $c(G-S) \leq c(C-S)$, then $G-S$ also has at most $|S|$ (connected components), because C is spanning subgraph of G.
- If $c(G-S) = |S|$. Then $C-S$ has exactly $|S|$ components, and these components are spanning subgraphs of the components of $G-S$. Thus C includes a Hamilton path in each component of $G-S$.

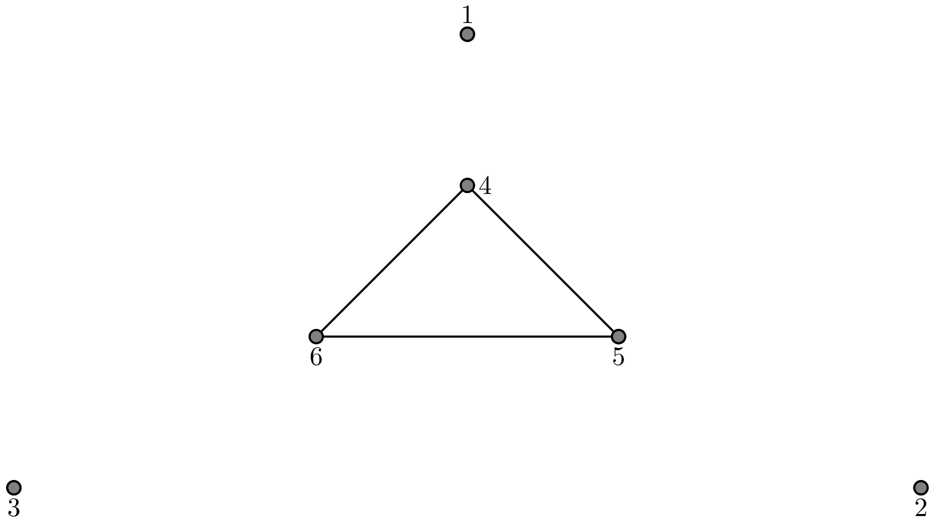
Example 2.4

$G = (\{1,2,3,4,5,6,a,b,c\},$
$\{\{1,a\},\{1,b\},\{1,c\},\{a,2\},\{a,b\},\{a,c\},\{a,3\},\{a,4\},\{a,5\},\{a,6\},\{4,5\},$
$\{4,6\},\{5,6\},\{b,2\},\{b,c\},\{b,3\},\{b,4\},\{b,5\},\{b,6\},\{c,2\},\{c,3\},\{c,4\},\{c,5\},\{c,6\}\})$



G

$G-\{a,b,c\}=(\{1,2,3,4,5,6\},\{\{4,5\},\{4,6\},\{5,6\}\})$



$G-\{a,b,c\}$

Thus, for $S=\{a,b,c\}$*, we have* $c(G-S)=4>|S|$*. So, the graph* G *is nonhamiltonian.*

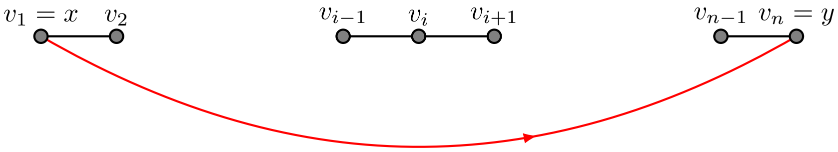
As a **sufficient condition**, we have the following theorem

Theorem 2.5 *(***Ore 1960***).*
Let $G=(V,E)$ *be a graph of order* $n\geq 3$*. If [for every nonadjacent vertices* $u\neq v$ *of* G*,* $d(u)+d(v)\geq n$*], then [* G *is hamiltonian].*

Proof.
We proceed by contradiction. Then we consider a non hamiltonian graph $G=(V,E)$ of order $n\geq 3$ such that: $[\forall x\neq y\in V$, if $(\{x,y\}\notin E)$ then $(d(x)+d(y)\geq n)]$, having the **maximum number of edges** (among such graphs).
As K_n is hamiltonian, then there are $x\neq y\in V:\ \{x,y\}\notin E$.
Consider the graph $\widetilde{G}=G+e$ where $e=\{x,y\}$ ("by adding the edge e"). By maximality, $\widetilde{G}$ is

hamiltonian. Thus, $\widetilde{G}$ has a Hamilton cycle C, and $\{x, y\}$ is necessarily an edge of C.

So, C can be written: $C = (v_1 = x, v_2, ..., v_i, ..., v_n = y, x)$.

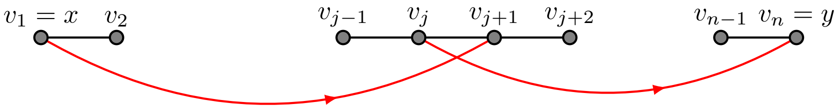


Let: $I = \{i \in \{1, ..., n-1\} : \{x, v_{i+1}\} \in E\}$
$J = \{i \in \{1, ..., n\} : \{y, v_i\} \in E\}$

Clearly, we have: $|I| = d_G(x)$ and $|J| = d_G(y)$. As ($n \notin I$ and $n \notin J$), then ($I \cup J \subseteq \{1, ..., n-1\}$, if and only if $|I \cup J| \leq n-1$). Moreover, $|I| + |J| = d_G(x) + d_G(y) \geq n$. Then $I \cap J \neq \emptyset$.

Consider then an element j of $I \cap J$. Then: $\{x, v_{j+1}\} \in E$ and $\{y, v_j\} \in E$.

In G, we have:



Then $C_1 = (v_1 = x, v_{j+1}, v_{j+2}, ..., v_n = y, v_j, v_{j-1}, ..., v_1 = x)$ is an Hamilton cycle of G; contradiction.

The following result is an immediate consequence of Theorem 2.5.

Theorem 2.6 *(***Dirac, 1952***)*

Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.

Bondy and Chvátal (1974) observed that the proof of theorem of Dirac can be modified to yield stronger sufficient condition than that obtained by Dirac.

Theorem 2.7

Let $G = (V, E)$ be a simple graph of order $n \geq 3$, and let u and v be nonadjacent vertices in G such that $d(u) + d(v) \geq n$, we have: G is hamiltonian if and only if $G + uv$ is hamiltonian.

This Theorem 2.7 motivates the following definition.

Definition 2.8

Let $G = (V, E)$ be a graph $|V| = n \geq 3$. The closure of G is the graph on V obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum at least n until no such pairs remains. The closure of G denotes $c(G)$.

Lemma 2.9

$c(G)$ is well defined.

Theorem 2.10

Let $G = (V, E)$ be a simple graph of order $n \geq 3$, G is hamiltonian if and only if $c(G)$ is hamiltonian.

Corollary 2.11

Let $G = (V, E)$ be a simple graph of order $n \geq 3$, If $c(G)$ is complete, then G is hamiltonian.

A more general condition than that of Dirac was obtained by Chvátal (1972).

Theorem 2.12

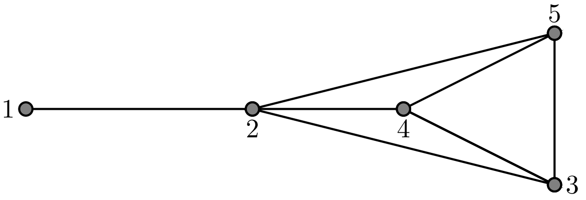
Let $G = (V, E)$ be a simple graph of order $n \geq 3$ with increasing degree sequence $(d_1, d_2, ..., d_n)$. Suppose that there is no value of $m \leq \frac{n}{2}$ for which $d_m \leq m$ and $d_{n-m} < n - m$. Then G is hamiltonian.

Definition 2.13

1. *A sequence of real numbers $(p_1, p_2, ..., p_n)$ is said majorised by another such sequence $(q_1, q_2, ..., q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$.*
2. *A graph G is degree-majorised by a graph H if $|V(G)| = |V(H)|$ and the nondecreasing degree sequence of G is majorised by that of H.*

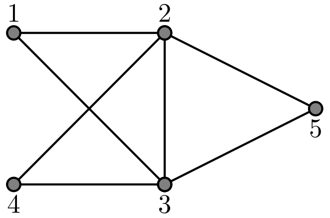
Example 2.14

1. *The 5-cycle is degree majorised by $K_{2,3}$ because $(2, 2, 2, 2, 2)$ is majorised by $(2, 2, 2, 3, 3)$.*
2. $C_{1,5} = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\})$



$C_{1,5}$

3. $C_{2,5} = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\})$

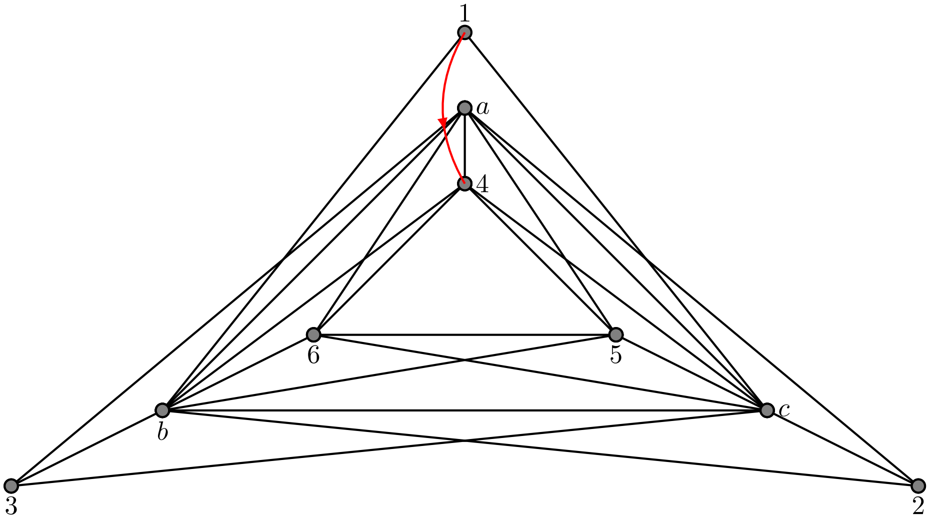


$C_{2,5}$

2.3 Exercises:

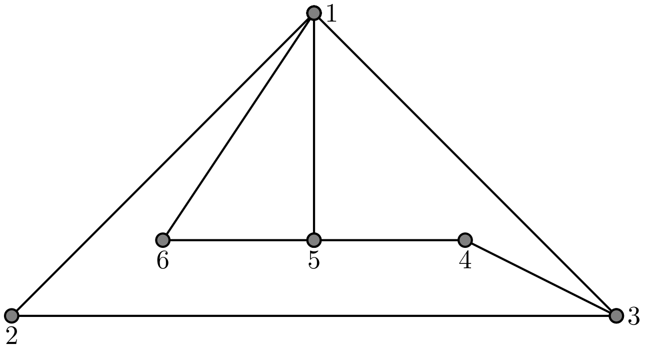
Prove the following assertions:

1. $G = (\{1,2,3,4,5,6,a,b,c\}$, $\{\{1,4\},\{1,b\},\{1,c\},\{a,2\},\{a,b\},\{a,c\},\{a,3\},\{a,4\},\{a,5\},\{a,6\},\{4,5\}$, $\{4,6\},\{5,6\},\{b,2\},\{b,c\},\{b,3\},\{b,4\},\{b,5\},\{b,6\},\{c,2\},\{c,3\},\{c,4\},\{c,5\},\{c,6\}\})$, is a hamiltonian graph.



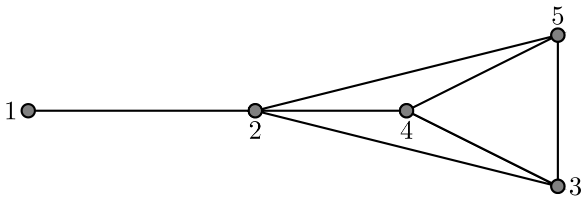
G

2. $G = (\{1,2,3,4,5,6\},\{\{1,2\},\{1,3\},\{1,5\},\{1,6\},\{2,3\},\{3,4\},\{4,5\},\{5,6\}\})$ is a hamiltonian graph.



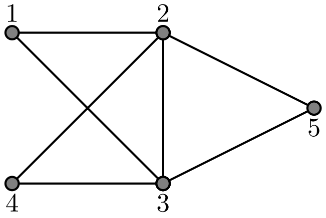
G

3. $C_{1,5} = (\{1,2,3,4,5\},\{\{1,2\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\})$ is a non hamiltonian graph.



$C_{1,5}$

4. $C_{2,5} = (\{1,2,3,4,5\}, \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\}\})$ is a non hamiltonian graph.



$C_{2,5}$

3 Exercises of Eulerian and Hamiltonian Graphs

Exercise 3.1

Classify the following graphs as eulerian, traversable, or neither.

1. $G_1 = (\{1,2,3,4,5,6\}, \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,6\},\{3,6\},\{4,6\},\{5,6\}\})$.
2. $G_2 = (\{1,2,3,4,5,6,7,8\}, \{\{1,2\},\{2,3\},\{3,4\},\{4,1\},\{5,6\},\{6,7\},\{7,8\},\{8,5\}\})$.
3. $G_4 = (\{1,2,3,4,5,6,7,8,9\},$ $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{8,1\},\{8,2\},\{2,4\},\{4,6\},\{6,8\},\{2,9\},\{4,9\},\{6,9\},\{8$
4. $G_4 = (\{1,2,3,4,5,6\}, \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,1\},\{1,3\},\{3,5\},\{5,1\},\{2,4\},\{4,6\},\{6,2\}\}$

Exercise 3.2

Let G_1 and G_2 be two eulerian graphs with no vertices in common. Let v_1 be a vertex of G_1 and let v_2 be a vertex of G_2. Consider the graph $G_3 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\}\})$ what can be said about G_3?

Exercise 3.3

Determine what special property is exhibited by a connected graph G with exactly four odd vertices with at least 2 are non adjacent.
(Hint: Consider a graph $\widetilde{G} = G + e$ where $e = \{u, v\}$ with $u \neq v$ are two nonadjacent odd vertices).

Exercise 3.4

Give an example of a graph G of order 10 vertices witch is:

1. *Hamiltonian.*
2. *Not hamiltonian.*

Exercise 3.5

Given an odd integers $p \geq 3$, give an example of a graph G of order p such that: ($\forall x \in V(G),\ d_G(x) \geq \frac{p-1}{2}$) and G is non hamiltonian.

Exercise 3.6

Let $G = (V, E)$ be a graph of order $p \geq 2$, such that: ($\forall x \in V(G),\ d_G(x) \geq \frac{p-1}{2}$). Show that G is traceable.

Exercise 3.7

Given a graph $G = (V, E)$, the line-graph of G is a graph $L(G)$ which describes the adjacencies of the edges of G (i.e: $L(G) = (E, E(L(G)))$ where, for $e \neq e' \in E$, ($\{e, e'\} \in E(L(G))$ if and only if $|e \cap e'| = 1$)),

Show that if a graph G is eulerian, then $L(G)$ is hamiltonian.

Exercise 3.8

1. *Determine all the integers $p \geq 1$, for which:*
    - *(a) K_p is eulerian.*

   (b) K_p *is hamiltonian.*

2. *Determine all the integers* $p, q \geq 1$, *for which:*
   (a) $K_{p,q}$ *is eulerian.*

   (b) $K_{p,q}$ *is hamiltonian.*

Exercise 3.9
Let $G = (V, E)$ *be an* m*-regular graph of order* $n \geq 2m + 2$, *where* $m \geq 1$. *Prove that* $\overline{G}$ *the complement of* G *is hamiltonian.*

Exercise 3.10
Let $G = (V_1 \cup V_2, E)$ *be a bipartite graph.*

1. *Prove that if* G *is hamiltonian, then* $|V_1| = |V_2|$.
2. *Prove that if* $|V_1| = |V_2| = n \geq 2$ *and* $\delta(G) > \frac{n}{2}$, *then* G *is hamiltonian.*

Exercise 3.11
Let $G = (V, E)$ *be a connected graph. Show that there exists an eulerian graph* H *such that* $v(H) \leq v(G) + 1$ *and* G *is an induced subgraph of* H.

Exercise 3.12
Let $G = (V, E)$ *be a graph and let* $e \in E$. *Show that if there exist two different cycles* C_1 *and* C_2 *both containing* e, *then* G *has a cycle that does not contain* e.

Exercise 3.13
Let $G = (V, E)$ *be a graph of order* $n \geq 4$. *Assume that for all distinct vertices* u, v, w *of* G, *the induced subgraph* $G[\{u, v, w\}]$ *has at least two edges.*
Show that G *is hamiltonian.*

Exercise 3.14
Prove that the subgraph $\overline{C_n}$ *of* G *is hamiltonian for all* $n \geq 5$.