Eulerian Graphs-Hamiltonian Graphs

October 12, 2026

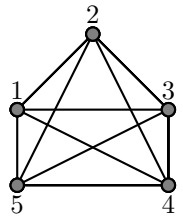
1 Eulerian Graphs

1.1 Definitions:

1. A *tour* of a connected graph G is a closed walk that traverses each edge of G at least once.
2. An *Euler tour* is a tour traversing each edge of G exactly once (i.e a closed Euler trail).
3. A graph is *eulerian* if it admits an Euler tour.

1.2 Examples:

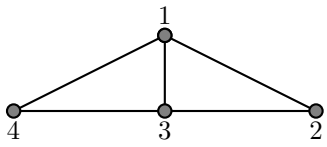
1. $G_1 \simeq K_5$ is an eulerian graph
   $G_1 = (\{1,2,3,4,5\}, \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\})$



$G_1 \simeq K_5$

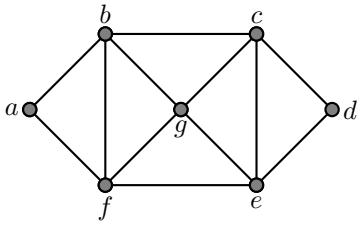
   Indeed: $C = (1,2,3,1,5,4,3,5,2,4,1)$ is an Euler tour.

2. $G_2 = (\{1,2,3,4\}, \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\}\})$ is non eulerian graph (You can verify).



G_2

3. $G_3 = (\{a,b,c,d,e,f,g\},$
   $\{\{a,b\},\{a,f\},\{b,c\},\{b,g\},\{b,f\},\{c,d\},\{c,e\},\{c,g\},\{d,e\},\{e,g\},\{e,f\},\{f,g\}\})$ is an eulerian graph.



G_3

Consider: $C = (a, b, c, d, e, c, g, f, e, g, b, f, a)$ is an Euler tour.

Remark 1.1.

Consider an eulerian graph G and an Euler tour $W = (u = \alpha_1, \alpha_2, ..., \alpha_p, u)$. Each time a vertex v occurs as an internal vertex of W, 2 edges incident with v are accounted for. But W traverses each edge exactly once. Then $\forall u \neq v$, $d_G(v)$ is even. Also, $d_G(u)$ is even, because W starts and ends at u. Thus, G is even.

Theorem 1.2. *(***Euler 1736***).*

A connected graph is eulerian if and only if it is even.

Proof.

1. " $\Rightarrow$ " If a connected graph is eulerian, then it is even by Remark.

2. " $\Leftarrow$ " Consider the converse statement.
   Assume that $G = (V, E)$ is a connected and even graph.
   $\rightarrow$ Select a **vertex** u of G and begin a **Trail** P at u. We continue this trail as long as possible (until we reach a vertex w such that the only edges incident with w already belong to P, hence, P cannot be continued, and we must stop).
   Fact 1: $w = u$ (i.e P is a closed trail).

   Indeed:

   If $w \neq u$, clearly an odd number of edges incident with w appears on P. As $d_G(w)$ is even, there is at least one edge incident with w that does not belong to P. Thus, P can be continued; contradiction.
   Fact 2: If $E(P) \subsetneq E(G)$, then there is one vertex (at least one) v on P that is incident with an edge $e \in E(G) \setminus E(P)$ (at least one edge e).

   Indeed:

   - If P contains all vertices of G, each $e \in E(G) \setminus E(P)$ satisfies the property.
   - If not, let X be the set of vertices of P. As $X \in \mathcal{P}(V) \setminus \{\emptyset, V\}$, then the edge cut of G, $\partial(X)$, associated with X is nonempty. Let $e = \{x, y\} \in \partial(X)$ where $x \in X$ (and $y \notin X$). So the vertex x and the edge e satisfy the property.

   N.B

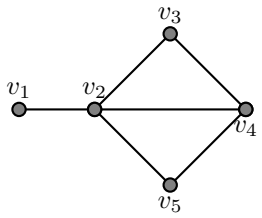
   - If $E(P) = E(G)$, then P is an Euler tour.
   - Assume that, $E(P) \subsetneq E(G)$.

$\rightarrow$ Now, remove the edges of P and consider the subgraphs
$H = G \setminus E(P) = (V, E(G) \setminus E(P))$. Clearly, every vertex in H has even degree. Let $H_1 = H[X_1]$ where X_1 is the connected component of H containing the vertex v (the vertex v is choose in Fact 2). We begin a trail P_1 in H_1 at v and continue this trail as long as possible. As before, P_1 is a closed trail.
Consider the closed trail $P = (\alpha_1 = u, \alpha_2, ..., \alpha_{\mathbf{i}} = \mathbf{v}, \alpha_{i+1}, ..., \alpha_p, u)$.
By inserting the closed trail P_1 at a place "$\alpha_i = v$", we obtain a closed trail C_1 of G, beginning and ending at u, which has more edges than P.

- If C_1 contains all the edges of G, then C_1 is an Euler tour, and G is an eulerian graph.
- If C_1 does not contain all the edges of G, then we may continue the above procedure until we finally obtain an Euler tour.

Definition 1.3. *(***Traversable graph***).*
If a connected graph G has a trail, which is not closed, and containing all edges of G, then G is called: "a traversable graph" and the trail is called: "an Euler trail".

Example 1.4.
$G = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}\})$.



G

The graph G is traversable and $P = (v_1, v_2, v_4, v_3, v_2, v_5, v_4)$ is an euler trail.

Theorem 1.5.
A connected graph G is traversable if and only if G has exactly two odd vertices. Furthermore, any euler trail of G begins at one of the odd vertices and ends at the other odd vertex.

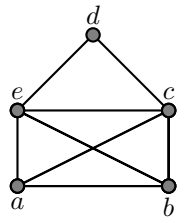
Proof.

- " $\Rightarrow$ " It is clear.
- " $\Leftarrow$ " Consider a connected graph G with exactly two odd vertices: a and b. Consider the graph $\widetilde{G} = (V(G) \cup \{z\}, E(G) \cup \{\{a, z\}, \{b, z\}\})$ where $z \notin V(G)$.
  By Theorem 1.2, $\widetilde{G}$ is an eulerian graph and then there is a closed trail
  $C = (\alpha_1 = a, \alpha_2 = z, \alpha_3, ..., \alpha_p = a)$. So $\alpha_3 = b$ and $P = (\alpha_3 = b, \alpha_4, ..., \alpha_p = a)$ is an euler trail of G, hence, G is a connected graph and traversable.

Remark 1.6.
An interesting property of eulerian and traversable graphs is that: Once the vertices have been drawn, we can draw the edges in one "continuous" motion (i.e the edges can be drawn without lifting the pencil from the paper). Note that the condition is: the number of odd vertices is zero or two.

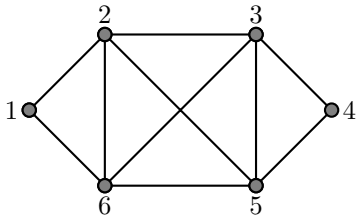
Example 1.7.

1. *"Envelope":* $G = (\{a,b,c,d,e\}, \{\{a,b\},\{a,c\},\{a,e\},\{b,c\},\{b,e\},\{c,d\},\{c,e\},\{d,e\}\})$ *We can draw this envelope without lifting the pencil from the paper, but for this, we must begin by one of* $\{a,b\}$ *and end at the other of* $\{a,b\}$.
(For example: $P = (a,c,e,b,a,e,d,c,b)$*).*



G

2. $G = (\{1,2,3,4,5,6\}, \{\{1,2\},\{1,6\},\{2,3\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{5,6\}\})$
G *is an eulerian graph because it connected and even.*



G

Find an Euler tour:

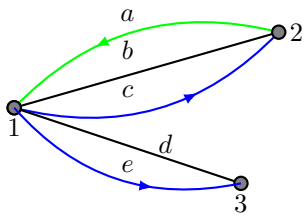
- *We begin at* $u = 1$*. we obtain the closed trail* $C_1 = (1,2,6,1)$.
- *The edge* $\{6,5\}$ *is not in* C_1 *(and* 6 *is a vertex of* C_1*).*
- *In* $G \setminus E(C_1)$*, we begin at* 6*, and we obtain* $C_2 = (6,5,4,3,2,5,3,6)$.
- *Insertion of* C_2 *in* C_1*, we obtain* $C_3 = (1,2,6,5,4,3,2,5,3,6,1)$ *is an Euler tour of* G.

Remarks 1.8. *(***The seven bridges of Königsberg***)*

1. *In* 1736*, Euler showed that it was impossible to cross each of the seven bridges of Königsberg once and only once during a walk through the town.*
Here is a plan of Königsberg and the river Pregel:

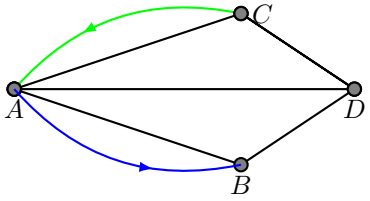
2. *For this we need the notion "multigraph".*

   - *In a multigraph, we can repeat some edges.*
     Example:
     $M = (\{1,2,3\}, \{a = \{1,2\}, b = \{1,2\}, c = \{1,2\}, d = \{1,3\}, e = \{1,3\}\})$.



   - *We can define the notions: walk, trail, connected multigraph.*

3. *For the problem of the seven bridges of Königsberg, consider the multigraph:*
   $(G = (\{A,B,C,D\}, \{\{A,B\}, \{A,B\}, \{A,C\}, \{A,C\}, \{A,D\}, \{C,D\}, \{B,D\}\}))$



   The multigraph G has 4 vertices of odd degree. So G is neither eulerian graph, nor traversable graph.

2 Hamiltonian Graphs

2.1 Definitions:

1. A path (resp. A cycle) of graph G is called a *Hamilton path* (resp. *Hamilton cycle*) if it contains every vertex of G.

2. A graph is *traceable* (resp. *hamiltonian*) if it contains a Hamilton path (resp. Hamilton cycle).

Remark 2.1.

1. *If a graph $G = (V, E)$ is hamiltonian, then:*

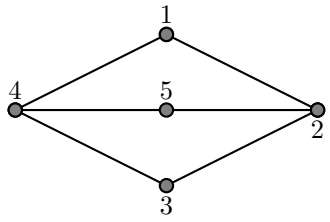
   (a) $|V| \geq 3$.

   (b) $\forall x \in V,\ d_G(x) \geq 2$.

   (c) G is connected.

   (d) G has no cut-vertex.

2. Consider the following graph: $G = (\{1,2,3,4,5\}, \{\{1,2\},\{1,4\},\{2,3\},\{2,5\},\{3,4\},\{4,5\}\})$.

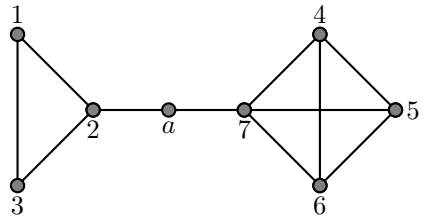


G

This graph G satisfies the 4 conditions on the first remark. However this graph is nonhamiltonian.

Example 2.2.

$G = (\{1,2,3,4,5,6,7,a\}, \{\{1,2\},\{2,3\},\{3,1\},\{2,a\},\{a,7\},\{4,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}\})$, *is non hamiltonian graph because the vertex a is a cut-vertex.*



G

2.2 Properties of hamiltonian graph:

As a **necessary condition** we have the following theorem

Theorem 2.3.

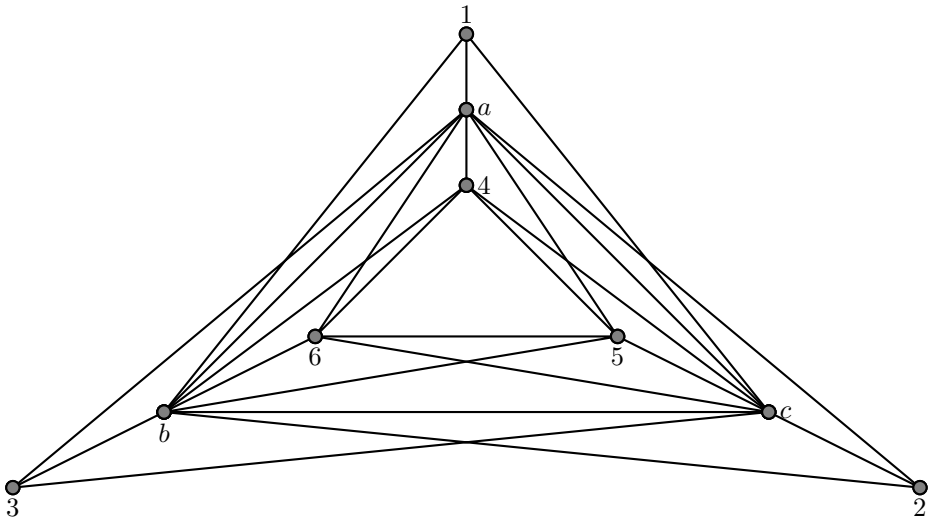
Given a hamiltonian graph $G = (V, E)$, then for every nonempty proper subset S of V, $c(G-S) \leq |S|$. Moreover, if equality holds, then each of the $|S|$ connected components of $G-S$ is traceable, and every Hamilton cycle of G includes a Hamilton path in each of these components.

Proof.

- Let C be a Hamilton cycle. Clearly $C - S$ has at most $|S|$ components and, since $c(G-S) \leq c(C-S)$, then $G-S$ also has at most $|S|$ (connected components), because C is spanning subgraph of G.
- If $c(G-S) = |S|$. Then $C - S$ has exactly $|S|$ components, and these components are spanning subgraphs of the components of $G - S$. Thus C includes a Hamilton path in each component of $G - S$.

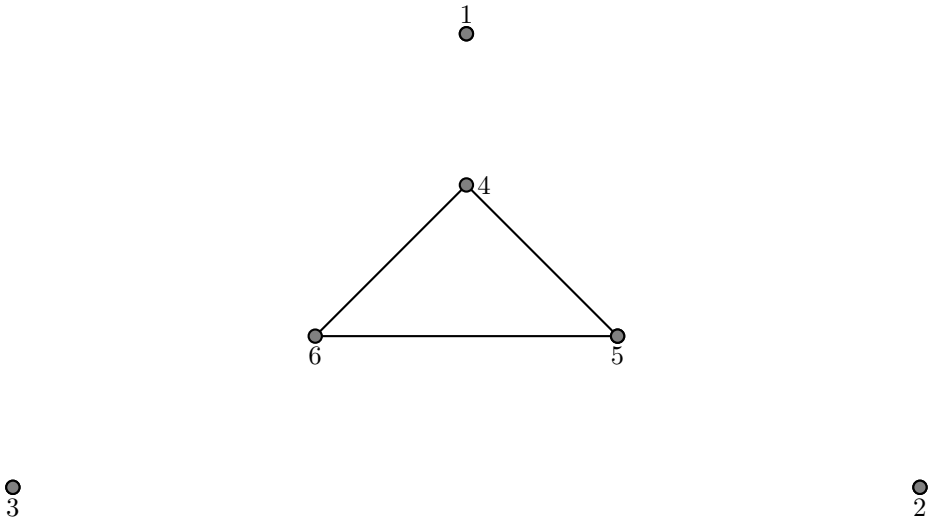
Example 2.4.

$G = (\{1,2,3,4,5,6,a,b,c\},$
$\{\{1,a\},\{1,b\},\{1,c\},\{a,2\},\{a,b\},\{a,c\},\{a,3\},\{a,4\},\{a,5\},\{a,6\},\{4,5\},$
$\{4,6\},\{5,6\},\{b,2\},\{b,c\},\{b,3\},\{b,4\},\{b,5\},\{b,6\},\{c,2\},\{c,3\},\{c,4\},\{c,5\},\{c,6\}\})$



G

$G - \{a, b, c\} = (\{1, 2, 3, 4, 5, 6\}, \{\{4, 5\}, \{4, 6\}, \{5, 6\}\})$



$G - \{a, b, c\}$

Thus, for $S = \{a, b, c\}$, we have $c(G - S) = 4 > |S|$. So, the graph G is nonhamiltonian.

As a **sufficient condition**, we have the following theorem

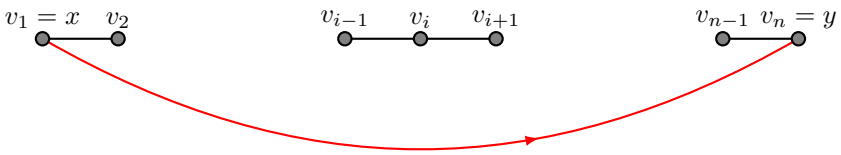
Theorem 2.5. *(***Ore 1960***).*

Let $G = (V, E)$ be a graph of order $n \geq 3$. If [for every nonadjacent vertices $u \neq v$ of G, $d(u) + d(v) \geq n$], then [G is hamiltonian].

Proof.

We proceed by contradiction. Then we consider a non hamiltonian graph $G = (V, E)$ of order $n \geq 3$ such that: $[\forall x \neq y \in V$, if $(\{x, y\} \notin E)$ then $(d(x) + d(y) \geq n)]$, having the **maximum number of edges** (among such graphs).
As K_n is hamiltonian, then there are $x \neq y \in V : \{x, y\} \notin E$.
Consider the graph $\widetilde{G} = G + e$ where $e = \{x, y\}$ ("by adding the edge e"). By maximality, $\widetilde{G}$ is hamiltonian. Thus, $\widetilde{G}$ has a Hamilton cycle C, and $\{x, y\}$ is necessarily an edge of C.

So, C can be written: $C = (v_1 = x, v_2, ..., v_i, ..., v_n = y, x)$.

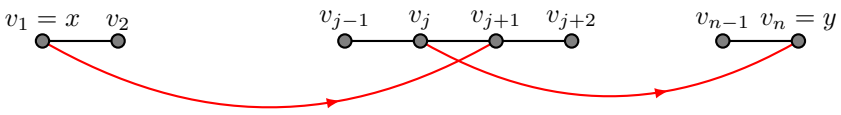


Let: $I = \{i \in \{1, ..., n-1\} : \{x, v_{i+1}\} \in E\}$
$J = \{i \in \{1, ..., n\} : \{y, v_i\} \in E\}$

Clearly, we have: $|I| = d_G(x)$ and $|J| = d_G(y)$. As ($n \notin I$ and $n \notin J$), then ($I \cup J \subseteq \{1, ..., n-1\}$, if and only if $|I \cup J| \leq n-1$). Moreover, $|I| + |J| = d_G(x) + d_G(y) \geq n$. Then $I \cap J \neq \emptyset$.

Consider then an element j of $I \cap J$. Then: $\{x, v_{j+1}\} \in E$ and $\{y, v_j\} \in E$.

In G, we have:



Then $C_1 = (v_1 = x, v_{j+1}, v_{j+2}, ..., v_n = y, v_j, v_{j-1}, ..., v_1 = x)$ is an Hamilton cycle of G; contradiction.

The following result is an immediate consequence of Theorem 2.5.

Theorem 2.6. *(***Dirac, 1952***)*
Let $G = (V, E)$ be a simple graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.

Bondy and Chvátal (1974) observed that the proof of theorem of Dirac can be modified to yield stronger sufficient condition than that obtained by Dirac.

Theorem 2.7.
Let $G = (V, E)$ be a simple graph of order $n \geq 3$, and let u and v be nonadjacent vertices in G such that $d(u) + d(v) \geq n$, we have: G is hamiltonian if and only if $G + uv$ is hamiltonian.

This Theorem 2.7 motivates the following definition.

Definition 2.8.
Let $G = (V, E)$ be a graph $|V| = n \geq 3$. The closure of G is the graph on V obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum at least n until no such pairs remains. The closure of G denotes $c(G)$.

Lemma 2.9.
$c(G)$ is well defined.

Theorem 2.10.
Let $G = (V, E)$ be a simple graph of order $n \geq 3$, G is hamiltonian if and only if $c(G)$ is hamiltonian.

Corollary 2.11.
Let $G = (V, E)$ be a simple graph of order $n \geq 3$, If $c(G)$ is complete, then G is hamiltonian.

A more general condition than that of Dirac was obtained by Chvátal (1972).

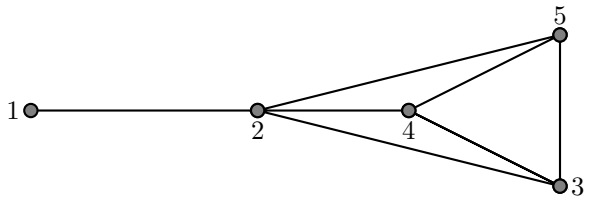
Theorem 2.12.
Let $G = (V, E)$ be a simple graph of order $n \geq 3$ with increasing degree sequence $(d_1, d_2, ..., d_n)$. Suppose that there is no value of $m \leq \frac{n}{2}$ for which $d_m \leq m$ and $d_{n-m} < n - m$. Then G is hamiltonian.

Definition 2.13.

1. *A sequence of real numbers $(p_1, p_2, ..., p_n)$ is said majorised by another such sequence $(q_1, q_2, ..., q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$.*
2. *A graph G is degree-majorised by a graph H if $|V(G)| = |V(H)|$ and the nondecreasing degree sequence of G is majorised by that of H.*

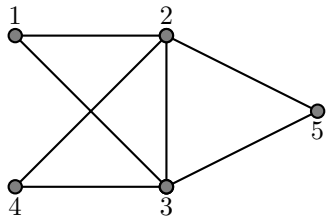
Example 2.14.

1. *The 5-cycle is degree majorised by $K_{2,3}$ because $(2, 2, 2, 2, 2)$ is majorised by $(2, 2, 2, 3, 3)$.*
2. $C_{1,5} = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\})$



$C_{1,5}$

3. $C_{2,5} = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\})$

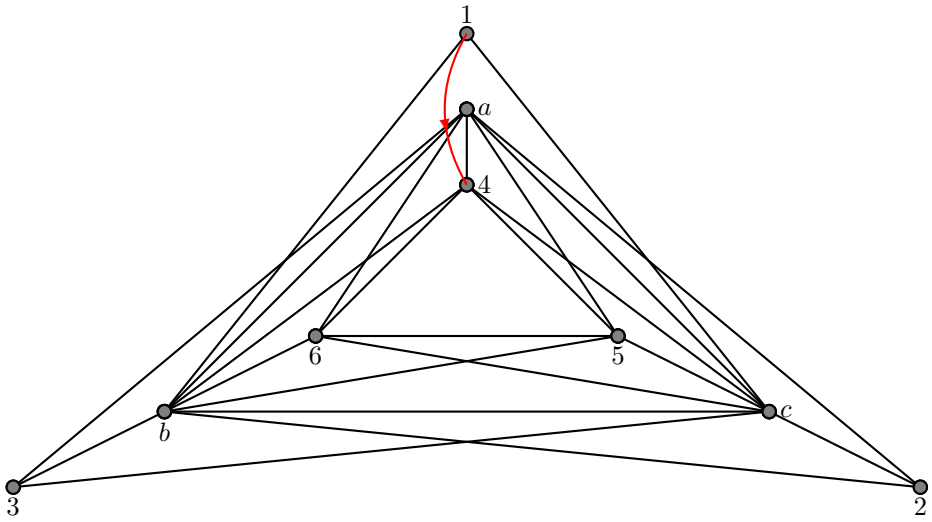


$C_{2,5}$

2.3 Exercises:

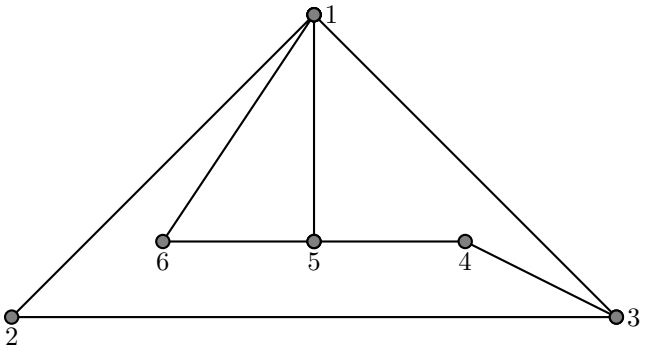
Prove the following assertions:

1. $G = (\{1, 2, 3, 4, 5, 6, a, b, c\},$
   $\{\{1, 4\}, \{1, b\}, \{1, c\}, \{a, 2\}, \{a, b\}, \{a, c\}, \{a, 3\}, \{a, 4\}, \{a, 5\}, \{a, 6\}, \{4, 5\},$
   $\{4, 6\}, \{5, 6\}, \{b, 2\}, \{b, c\}, \{b, 3\}, \{b, 4\}, \{b, 5\}, \{b, 6\}, \{c, 2\}, \{c, 3\}, \{c, 4\}, \{c, 5\}, \{c, 6\}\})$, is a hamiltonian graph.



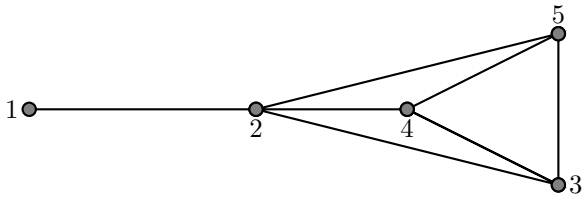
G

2. $G = (\{1,2,3,4,5,6\}, \{\{1,2\},\{1,3\},\{1,5\},\{1,6\},\{2,3\},\{3,4\},\{4,5\},\{5,6\}\})$ is a hamiltonian graph.



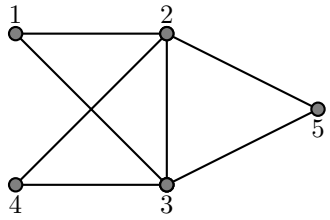
G

3. $C_{1,5} = (\{1,2,3,4,5\}, \{\{1,2\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\})$ is a non hamiltonian graph.



$C_{1,5}$

4. $C_{2,5} = (\{1,2,3,4,5\}, \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\}\})$ is a non hamiltonian graph.



$C_{2,5}$

3 Exercises with Answers

Exercise 1.

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.

1. Prove that if G is Hamiltonian, then $|V_1| = |V_2|$.
2. Prove that if $|V_1| = |V_2| = n \geq 2$ and $\delta(G) > \frac{n}{2}$, then G is Hamiltonian.

1. Prove that if G is Hamiltonian, then $|V_1| = |V_2|$.

Proof. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. By the definition of a bipartite graph, every edge in E has one endpoint in V_1 and the other in V_2. Therefore, any path or cycle in G must strictly alternate between vertices of V_1 and vertices of V_2.

Assume G is Hamiltonian. Then, G contains a Hamiltonian cycle C, which is a closed walk that visits every vertex of G exactly once. Let the sequence of vertices in the cycle be:

$$C = (v_1, v_2, v_3, \ldots, v_{2k}, v_1)$$

Since the vertices must alternate between the two independent sets, without loss of generality, all odd-indexed vertices $(v_1, v_3, \ldots, v_{2k-1})$ must belong to V_1, and all even-indexed vertices $(v_2, v_4, \ldots, v_{2k})$ must belong to V_2.

Because C visits every vertex exactly once and returns to the starting vertex, the cycle must contain an equal number of vertices from V_1 and V_2. Thus, the total number of vertices in V_1 visited by C equals the total number of vertices in V_2 visited by C. Since C visits all vertices in G, we conclude that:

$$|V_1| = |V_2|$$

□

2. Show that if $|V_1| = |V_2| = n \geq 2$ and $\delta(G) > n/2$, then G is Hamiltonian.

Proof. **Case 1.** For $n = 2$, if $|V_1| = |V_2| = 2$ and $\delta(G) > 2/2 = 1$, then $\delta(G) \geq 2$. The only bipartite graph on $2 + 2$ vertices satisfying this condition is the complete bipartite graph $K_{2,2}$, which is isomorphic to the cycle C_4. Therefore, G is Hamiltonian.

Case 2. For $n \geq 3$, we will prove the statement by contradiction using a variation of Ore's Theorem adapted for bipartite graphs.

Assume G is a bipartite graph with $|V_1| = |V_2| = n \geq 3$ and $\delta(G) > n/2$ that is **not** Hamiltonian. Edges may be added to G between V_1 and V_2 until the addition of any single further edge makes the graph Hamiltonian. Let this maximal non-Hamiltonian bipartite graph be $G' = (V_1 \cup V_2, E')$. Note that the minimum degree condition still holds since $\delta(G') \geq \delta(G) > n/2$.

Since G' is maximal non-Hamiltonian, it cannot be the complete bipartite graph $K_{n,n}$ (which is Hamiltonian). Thus, there must exist two non-adjacent vertices $u \in V_1$ and $v \in V_2$ in G' ($u \not\sim v$). By the maximality of G', adding the edge uv creates a Hamiltonian cycle, which implies that there already exists a Hamiltonian path P in G' starting at u and ending at v:

$$P = (u = x_1, y_1, x_2, y_2, \ldots, x_n, y_n = v)$$

where $x_i \in V_1$ and $y_i \in V_2$ for all $i \in \{1, \ldots, n\}$.

We define two sets of indices based on the neighbors of u and v:

$$S = \{i \mid y_i \in N_{G'}(u)\}$$
$$T = \{i \mid x_i \in N_{G'}(v)\}$$

Since $u \not\sim v$ and $y_n = v$, we know that $n \notin S$, meaning $S \subset \{1, 2, \ldots, n-1\}$. Similarly, since $x_1 = u$, we know that $1 \notin T$, meaning $T \subset \{2, 3, \ldots, n\}$. Both S and T are subsets of $\{1, 2, \ldots, n\}$.

The sizes of these sets correspond to the degrees of the vertices in G':

$$|S| = d_{G'}(u) \quad \text{and} \quad |T| = d_{G'}(v)$$

By the minimum degree condition, we have:

$$|S| + |T| = d_{G'}(u) + d_{G'}(v) \geq \delta(G') + \delta(G') > \frac{n}{2} + \frac{n}{2} = n$$

Hence, we obtain:

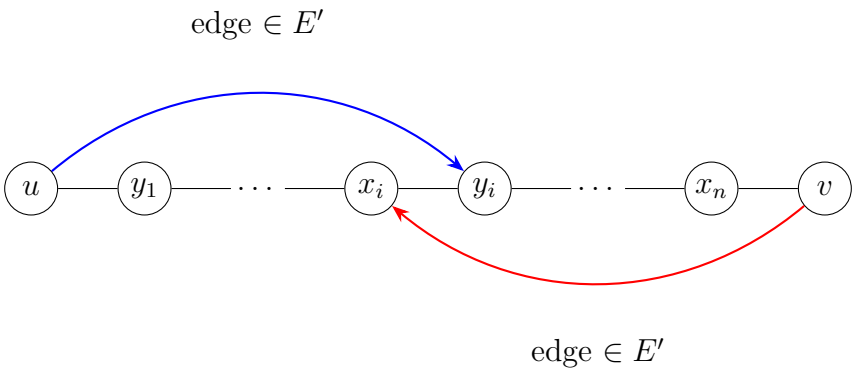
$$|S| + |T| \geq n + 1$$

Since S and T are both subsets of the set $\{1, 2, \ldots, n\}$ of size n, by the Pigeonhole Principle, they must share a common index. That is, there exists some index $i \in \{1, 2, \ldots, n\}$ such that $i \in S$ and $i \in T$.

Because $i \in S$ and $i \in T$, we have:

$$y_i \in N_{G'}(u) \quad \text{and} \quad x_i \in N_{G'}(v)$$

Structural Explanation of the Cycle: The existence of this common index i allows us to close a cycle without using the non-existent edge uv. We achieve this by splitting the Hamiltonian path P into segments and utilizing the cross-edges (u, y_i) and (x_i, v) as shown in the diagram below:



To traverse this new Hamiltonian cycle C, we follow these steps:

1. Start at u, and walk along the original path P up to the vertex x_i: $(u, y_1, x_2, \ldots, x_i)$.
2. Use the edge (x_i, v) to jump directly to the end of the path.
3. Walk **backward** along the path P from v down to y_i: $(v, x_n, y_{n-1}, \ldots, y_i)$.

4. Finally, use the edge (y_i, u) to jump back to the starting vertex u.

Formally, the sequence is written as:

$$C = (u, y_1, x_2, y_2, \ldots, x_i, v, x_n, y_{n-1}, \ldots, y_i, u)$$

This cycle contains every vertex of G' exactly once, which contradicts our fundamental assumption that G' is non-Hamiltonian. Therefore, the original graph G must be Hamiltonian. □

Exercise 2.

Let $G = (V, E)$ be a graph of order $n \geq 4$. Assume that for all distinct vertices u, v, w of G, the induced subgraph $G[\{u, v, w\}]$ has at least two edges. Show that G is Hamiltonian.

Show that G is Hamiltonian.

Proof. Let $G = (V, E)$ be a graph of order $n \geq 4$. The problem states that for any three distinct vertices u, v, w, the induced subgraph $G[\{u, v, w\}]$ contains at least two edges.

First, let us analyze the implication of this condition on the degrees of non-adjacent vertices. Let u and v be any two distinct, non-adjacent vertices in G ($uv \notin E$). Let w be any third vertex in $V \setminus \{u, v\}$.

Consider the induced subgraph $G[\{u, v, w\}]$. The only possible edges in this subgraph are uv, uw, and vw. Since $uv \notin E$ by assumption, the remaining two pairs must be edges to satisfy the condition of having at least two edges. Thus, we must have:

$$uw \in E \quad \text{and} \quad vw \in E$$

Since this holds for **every** choice of $w \in V \setminus \{u, v\}$, it implies that every other vertex in the graph is a common neighbor of both u and v.

Consequently, we can determine the degrees of u and v:

$$d(u) = n - 2 \quad \text{and} \quad d(v) = n - 2$$

Therefore, for any pair of non-adjacent vertices u and v, the sum of their degrees is:

$$d(u) + d(v) = (n - 2) + (n - 2) = 2n - 4$$

We can now apply **Ore's Theorem**, which states that a graph of order $n \geq 3$ is Hamiltonian if $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u, v.

Let us verify if this inequality holds for our graph:

$$2n - 4 \geq n \iff n \geq 4$$

Since the problem explicitly specifies that $n \geq 4$, the condition $d(u) + d(v) \geq n$ is always satisfied for every pair of non-adjacent vertices. By Ore's Theorem, G is Hamiltonian. □

Exercise 3. Dirac's Theorem: Statement and Proof

Theorem 3.1 (Dirac)**.** *Let F be a graph of order $m \geq 3$. If $\delta(F) \geq m/2$, then F is Hamiltonian.*

Proof. Suppose that F is not Hamiltonian, and choose a non-Hamiltonian supergraph H on the same vertex set with as many edges as possible. Since the complete graph K_m is Hamiltonian, H must have two non-adjacent vertices x and y. By the maximality of H, the graph $H + xy$ is Hamiltonian; deleting the edge xy from a Hamiltonian cycle gives a Hamiltonian path

$$v_1 v_2 \cdots v_m, \qquad \text{where } v_1 = x \text{ and } v_m = y,$$

in H. Consider the following index sets:

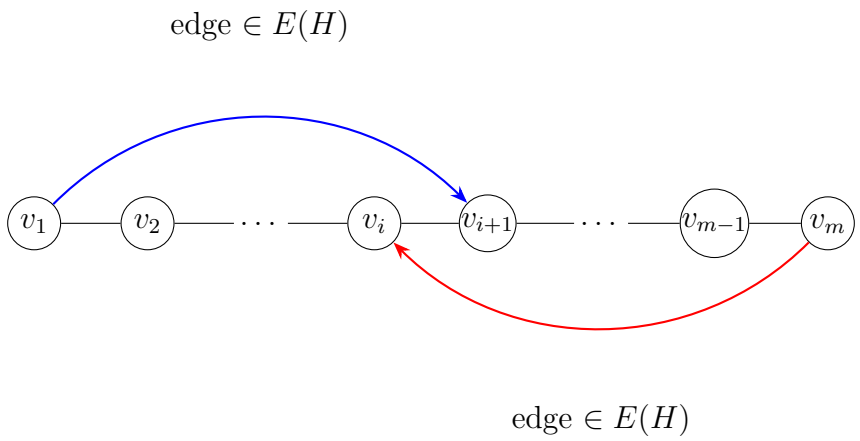
$$A = \{i \in \{1, \ldots, m-1\} \mid v_1 v_{i+1} \in E(H)\}, \qquad B = \{i \in \{1, \ldots, m-1\} \mid v_i v_m \in E(H)\}.$$

We have:

$$|A| + |B| = d_H(v_1) + d_H(v_m) \geq \delta(H) + \delta(H) \geq \delta(F) + \delta(F) \geq \frac{m}{2} + \frac{m}{2} = m.$$

Since both sets A and B are contained in $\{1, \ldots, m-1\}$, which has a cardinality of $m-1$, by the Pigeonhole Principle, they must intersect.

Structural Explanation of the Cycle: The intersection implies that there exists at least one index $i \in A \cap B$. This shared index means that v_1 is connected to v_{i+1}, and v_m is connected to v_i. We can visualize these connections and the structure of the resulting cycle in the diagram below:



To traverse this new Hamiltonian cycle, we reconstruct the path by breaking the internal edge (v_i, v_{i+1}) and following this specific sequence:

1. Start at v_1 and follow the original path forward up to v_i: $(v_1, v_2, \ldots, v_i)$.
2. Jump from v_i directly to the end vertex v_m using the edge (v_i, v_m).
3. Walk **backward** along the path from v_m down to v_{i+1}: $(v_m, v_{m-1}, \ldots, v_{i+1})$.

4. Use the edge (v_{i+1}, v_1) to jump from v_{i+1} back to the starting vertex v_1.

Formally, this produces the closed cycle:

$$v_1 v_2 \cdots v_i v_m v_{m-1} \cdots v_{i+1} v_1.$$

Note that $i \neq m-1$ because $v_1 v_m \notin E(H)$. This cycle visits every vertex of H exactly once, which directly contradicts our choice of H being a non-Hamiltonian graph. Therefore, the original graph F must be Hamiltonian. □

4 Exercises of Eulerian and Hamiltonian Graphs

Exercise 4.1.

Classify the following graphs as eulerian, traversable, or neither.

1. $G_1 = (\{1,2,3,4,5,6\}, \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,6\},\{3,6\},\{4,6\},\{5,6\}\})$.
2. $G_2 = (\{1,2,3,4,5,6,7,8\}, \{\{1,2\},\{2,3\},\{3,4\},\{4,1\},\{5,6\},\{6,7\},\{7,8\},\{8,5\}\})$.
3. $G_4 = (\{1,2,3,4,5,6,7,8,9\},$
   $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{8,1\},\{8,2\},\{2,4\},\{4,6\},\{6,8\},\{2,9\},\{4,9\},$
   $\{6,9\},\{8,9\}\})$.
4. $G_4 = (\{1,2,3,4,5,6\},$
   $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,1\},\{1,3\},\{3,5\},\{5,1\},\{2,4\},\{4,6\},\{6,2\}\})$.

Exercise 4.2.

Let G_1 and G_2 be two eulerian graphs with no vertices in common. Let v_1 be a vertex of G_1 and let v_2 be a vertex of G_2. Consider the graph
$G_3 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{\{v_1, v_2\}\})$ what can be said about G_3?

Exercise 4.3.

Determine what special property is exhibited by a connected graph G with exactly four odd vertices with at least 2 are non adjacent.
(Hint: Consider a graph $\widetilde{G} = G + e$ where $e = \{u, v\}$ with $u \neq v$ are two nonadjacent odd vertices).

Exercise 4.4.

Give an example of a graph G of order 10 vertices witch is:

1. *Hamiltonian.*
2. *Not hamiltonian.*

Exercise 4.5.

Given an odd integers $p \geq 3$, give an example of a graph G of order p such that: ($\forall x \in V(G)$, $d_G(x) \geq \frac{p-1}{2}$) and G is non hamiltonian.

Exercise 4.6.

Let $G = (V, E)$ be a graph of order $p \geq 2$, such that: ($\forall x \in V(G)$, $d_G(x) \geq \frac{p-1}{2}$). Show that G is traceable.

Exercise 4.7.

Given a graph $G = (V, E)$, the line-graph of G is a graph $L(G)$ which describes the adjacencies of the edges of G (i.e: $L(G) = (E, E(L(G)))$ where, for $e \neq e' \in E$, ($\{e, e'\} \in E(L(G))$ if and only if $|e \cap e'| = 1$)),

Show that if a graph G is eulerian, then $L(G)$ is hamiltonian.

Exercise 4.8.

1. *Determine all the integers $p \geq 1$, for which:*
   (a) *K_p is eulerian.*

(b) K_p is hamiltonian.

2. *Determine all the integers $p, q \geq 1$, for which:*

   (a) $K_{p,q}$ is eulerian.

   (b) $K_{p,q}$ is hamiltonian.

Exercise 4.9.
Let $G = (V, E)$ be an m-regular graph of order $n \geq 2m + 2$, where $m \geq 1$. Prove that $\overline{G}$ the complement of G is hamiltonian.

Exercise 4.10.
Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.

1. *Prove that if G is hamiltonian, then $|V_1| = |V_2|$.*

2. *Prove that if $|V_1| = |V_2| = n \geq 2$ and $\delta(G) > \frac{n}{2}$, then G is hamiltonian.*

Exercise 4.11.
Let $G = (V, E)$ be a connected graph. Show that there exists an eulerian graph H such that $v(H) \leq v(G) + 1$ and G is an induced subgraph of H.

Exercise 4.12.
Let $G = (V, E)$ be a graph and let $e \in E$. Show that if there exist two different cycles C_1 and C_2 both containing e, then G has a cycle that does not contain e.

Exercise 4.13.
Let $G = (V, E)$ be a graph of order $n \geq 4$. Assume that for all distinct vertices u, v, w of G, the induced subgraph $G[\{u, v, w\}]$ has at least two edges.
Show that G is hamiltonian.

Exercise 4.14.
Prove that the subgraph $\overline{C_n}$ of G is hamiltonian for all $n \geq 5$.