3.5.6 Scheffé's Method for Comparing All Contrasts

In many situations, experimenters may not know in advance which contrasts they wish to compare, or they may be interested in more than $(a-1)$ possible comparisons. In many exploratory experiments, the comparisons of interest are discovered only after preliminary examination of the data. Scheffé (1953) has proposed a method for comparing any and all possible contrasts between treatment means. In the Scheffé method, the type I error is at most for any of the possible comparisons

▪ The Steps for conducting Scheffé method.

Suppose that a set of m contrasts in the treatment means

1- Form the contrast number u, $u = 1,2,\cdots m$

$$\Gamma_u = c_{1u}\mu_1 + c_{2u}\mu_2 + \cdots c_{au}\mu_a = \sum_{i=1}^{a} c_{iu}\mu_i$$

2- Compute the point estimate to contrast Γ_u .

$$\hat{\Gamma}_u = C_u = \sum_{i=1}^{a} c_{iu}\bar{y}_{i\cdot}$$

where C_u is unbiased estimate to Γ_u

3- The standard error of this contrast is

$$S_{C_u} = \sqrt{\frac{MSE}{n}\sum_{i=1}^{a} c_{iu}^2} \quad \text{Balance} \qquad\qquad S_{C_u} = \sqrt{MSE\sum_{i=1}^{a}\left(\frac{c_{iu}^2}{n_i}\right)} \quad \text{Unbalance}$$

4- The critical value against which C_u should be compared is

$$S_{\alpha,u} = S_{C_u}\sqrt{(a-1)F_{\alpha,a-1,N-a}}$$

5- If $|C_u| > S_{\alpha,u}$ the null hypothesis $H_0\!: \Gamma_u = 0$ is rejected

❑ **Example 1 :** The response time in milliseconds was determined for three different types of circuits that could be used in an automatic valve shutoff mechanism. The results from a completely randomized experiment are shown in the following table.

Circuit Type	Response Time				
1	9	12	10	8	15
2	20	21	23	17	30
3	6	5	8	16	7

a) Use Scheffé method to compare type 2 with the other two.

b) Verify the results using the R program.

The Answer of Example 1 (a)

The contrast is: $\mu_2 = \frac{\mu_1+\mu_3}{2} \rightarrow \Gamma = -\mu_1 + 2\mu_2 - \mu_3 = 0, \quad c_i = (-1), (+2), (-1)$

Circuite Type	response Time					$y_{i\cdot}$	$\bar{y}_{i\cdot}$	c_i
1	9	12	10	8	15	54	10.8	-1
2	20	21	23	17	30	111	22.2	+2
3	6	5	8	16	7	42	8.4	-1

- **Create ANOVA table**

S.O.V	SS	df	MS	F	F crit	P-value
Circuite Type	543.6	2	271.8	16.0828	3.8853	0.0004
Error	202.8	12	16.9			
Total	746.4	14				

- **Hypothesis H_0, H_1**

  $H_0\!: \Gamma = 0 \quad vs\ H_1\!: \Gamma \neq 0$

- **Point estimate of Contrast Γ**

  $C = \sum_{i=1}^{3} c_i \bar{y}_{i\cdot} = (-1)\bar{y}_{1\cdot} + (2)\bar{y}_{2\cdot} + (-1)\bar{y}_3 = (-1)(10.8) + (2)(22.2) + (-1)(8.4) = 25.2.$

- **The standard error of C:** $S_C = \sqrt{\frac{MSE}{n}\sum_{i=1}^{3} c_i^2} = \sqrt{\frac{16.9}{5}(6)} = \sqrt{20.2} = 4.494$

- **Critical value:** $S_\alpha = S_C\sqrt{(a-1)F_{\alpha,a-1,N-a}} = 4.494\sqrt{(2)(3.8853)} = 12.527$

- **Decision :** $C = 25.2 > S_\alpha = 12.527$**, The null hypothesis** $H_0\!: \Gamma = 0$ **is rejected**

The Answer of Example 1 (b)

```
# Input data
                  y=c(9,12,10,8,15,20,21,23,17,30,6,5,8,16,7)
                  Type = factor(rep(c("Type1","Type2","Type3"),each=5))
                  data1 = data.frame(y,Type)
#Run Model
                  Model = aov(y~Type, data = data1); summary(Model)


# Scheffé’s Method to contrast (mean of Type2- mean of Type1 & Type3)
                  install.packages("DescTools")
                  library(DescTools)


         # Contrast: (mean of Type2- mean of Type1 & Type3)
         # Vector: -Type1 + 2Type2 -Type3
                  c.i <- matrix(c(-1,2,-1), ncol = 1)
                  Cont1 = ScheffeTest(Model, contrasts = c.i)
                  print(Cont1)
```

Output

```
> Model = aov(y~Type, data = data1); summary(Model)
            Df Sum Sq Mean Sq F value   Pr(>F)    
Type         2  543.6   271.8   16.08 0.000402 ***
Residuals   12  202.8    16.9                     
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
> Cont1 = ScheffeTest(Model, contrasts = c.i)
>
> print(Cont1)

  Posthoc multiple comparisons of means: Scheffe Test 
    95% family-wise confidence level

$Type
                  diff   lwr.ci   upr.ci    pval    
Type2-Type1,Type3 25.2 12.64661 37.75339 0.00045 ***

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We note that p value = 0.00045 < 0.05 we reject null hypothesis H_0: $\Gamma = 0$

3.5.7 Comparing Pairs of Treatment Means

In many practical situations, we will wish to compare only ***pairs of means*** $(\mu_i - \mu_j)$. Frequently, we can determine which means differ by testing the differences between all pairs $[\frac{a(a-1)}{2}]$ of treatment means. Suppose that we are interested in comparing all pairs of a treatment means and that the null hypotheses that we wish to test are $H_0: \mu_i = \mu_j,\ for\ all\ i \neq j$. We now present two popular methods for making such comparisons.

- **Tukey's Test.**

When the null hypothesis of equal treatment means $H_0: \mu_1 = \mu_2 = \cdots = \mu_a$, is rejected, we wish to test all pairwise mean comparisons $H_0: \mu_i = \mu_j$ vs $H_1: \mu_i \neq \mu_j$

The steps of **Tukey** procedure as follows

1. At significant level α , determine critical value $q_\alpha(a, f)$ from **Appendix Table VII** , where $a = number\ of\ treatments, f = error\ degrees\ of\ freedom$
2. Compute T_α, which given by

$$T_\alpha = q_\alpha(a, f)\sqrt{\frac{MSE}{n}} \qquad (3.35)$$

3. If the absolute difference $|\bar{y}_{i.} - \bar{y}_{j.}| > T_\alpha$ null hypothesis $H_0: \mu_i = \mu_j$ is rejected and it concluded that existing significant difference between mean μ_i and mean μ_j

❑ **Example 2:** Use the data in example 1, to conduct comparing pairs of treatment means by Tukey's test.

a) Manually **b) R program**

====================================

The Answer

S.O.V	*SS*	*df*	*MS*	*F*	*F crit*	*P-value*
Circuite Type	543.6	2	271.8	16.0828	3.8853	0.0004
Error	202.8	12	16.9			
Total	746.4	14				

a) Manually

- Compute T_α, which given by:

$$T_\alpha = q_\alpha(a,f)\sqrt{\frac{MSE}{n}} = q_{0.05}(3,12)\sqrt{\frac{16.9}{5}} = 3.77(1.8385) = 6.93$$

$\bar{y}_{i\cdot}$	$\bar{y}_{j\cdot}$	$\lvert\bar{y}_{i\cdot} - \bar{y}_{j\cdot}\rvert$	T_α	H_0	Decision
10.8	**22.2**	**11.4**	**6.93**	$H_0: \mu_1 = \mu_2$	H_0 is rejected
10.8	**8.4**	**2.4**	**6.93**	$H_0: \mu_1 = \mu_3$	H_0 is not rejected
22.2	**8.4**	**13.8**	**6.93**	$H_0: \mu_2 = 3$	H_0 is rejected

b) R program

```
# Input libraries
        library(car);   library(multcomp);   library(DescTools)
# Input data
                y=c(9,12,10,8,15,20,21,23,17,30,6,5,8,16,7)
                Type = factor(rep(c("Type1","Type2","Type3"),each=5))
                data1 = data.frame(y,Type)
#Run Model
                Model = aov(y~Type, data = data1); summary(Model)
# TukeyHSD Method to pairwise comparisons
                TukeyHSD(Model)
```

```
> TukeyHSD(Model)
  Tukey multiple comparisons of means
    95% family-wise confidence level

Fit: aov(formula = y ~ Type, data = data1)

$Type
               diff        lwr       upr     p adj
Type2-Type1  11.4   4.463555 18.336445 0.0023656
Type3-Type1  -2.4  -9.336445  4.536445 0.6367043
Type3-Type2 -13.8 -20.736445 -6.863555 0.0005042
```

- **The Fisher Least Significant Difference (*LSD*) Method.**

The steps of Least Significant Difference (*LSD*) for testing all pairwise mean comparisons $H_0: \mu_i = \mu_j$ vs $H_1: \mu_i \neq \mu_j$ as follows

1. At significant level α , compute the value (*LSD*), which given by

$$LSD = t_{\frac{\alpha}{2},N-a}\sqrt{\frac{2MSE}{n}} \quad \text{or} \quad LSD = t_{\frac{\alpha}{2},N-a}\sqrt{MSE\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$$

2. If the absolute difference $|\bar{y}_{i.} - \bar{y}_{j.}| > LSD$ null hypothesis $H_0: \mu_i = \mu_j$ is rejected and it concluded that existing significant difference between mean μ_i and mean μ_j

❑ **Example 3:** Use the data in example 1, to conduct comparing pairs of treatment means by *LSD*.

a) Manually **b) R program**

==

The Answer

a) Manually

Compute LSD, which given by:

$$LSD = t_{\frac{\alpha}{2},N-a}\sqrt{\frac{2MSE}{n}} = t_{\frac{0.05}{2},12}\sqrt{\frac{2\times 16.9}{5}} = 2.179(2.6) = 5.665$$

$\bar{y}_{i\cdot}$	$\bar{y}_{j\cdot}$	$\lvert\bar{y}_{i\cdot} - \bar{y}_{j\cdot}\rvert$	LSD	H_0	Decision
10.8	22.2	11.4	5.665	$H_0: \mu_1 = \mu_2$	H_0 is rejected
10.8	8.4	2.4	5.665	$H_0: \mu_1 = \mu_3$	H_0 is not rejected
22.2	8.4	13.8	5.665	$H_0: \mu_2 = 3$	H_0 is rejected

b) R program

```
library(car);   library(multcomp);   library(DescTools); library(agricolae)
                  y=c(9,12,10,8,15,20,21,23,17,30,6,5,8,16,7)
                  Type = factor(rep(c("Type1","Type2","Type3"),each=5))
                  data1 = data.frame(y,Type)
                  Model = aov(y~Type, data = data1); summary(Model)
                  print(LSD.test(Model,trt = "Type"))
```

The output

```
> print(LSD.test(Model,trt = "Type"))
$statistics
  MSerror Df Mean       CV  t.value      LSD
     16.9 12 13.8 29.78957 2.178813 5.664913

$parameters
        test p.ajusted name.t ntr alpha
  Fisher-LSD      none   Type   3  0.05

$means
         y      std r       se       LCL     UCL Min Max Q25 Q50 Q75
Type1 10.8 2.774887 5 1.838478  6.794301 14.8057   8  15   9  10  12
Type2 22.2 4.868265 5 1.838478 18.194301 26.2057  17  30  20  21  23
Type3  8.4 4.393177 5 1.838478  4.394301 12.4057   5  16   6   7   8

$comparison
NULL

$groups
         y groups
Type2 22.2      a
Type1 10.8      b
Type3  8.4      b
```

Means with the same letters are not significantly different; therefore, there is a significant difference between the mean of type 2 and both types 1 and 3, while there is no significant difference between the means of type 1 and 3.

3.5.8 Comparing Treatment Means with a Control

In many experiments, one of the treatments is a control, and the analyst is interested in comparing each of the other $a-1$ treatment means with the control. A procedure for making these comparisons has been developed by **Dunnett** (1964). Suppose that treatment a is the control and we wish to test the hypotheses

$$H_0: \mu_i = \mu_a \quad \text{vs} \quad H_1: \mu_i \neq \mu_a$$

The null hypothesis $H_0: \mu_i = \mu_a$ is rejected using a type I error rate α if

$$|\bar{y}_{i\cdot} - \bar{y}_{a\cdot}| > d_\alpha(a-1, f)\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_a}\right)}$$

where the constant $d_\alpha(a-1, f)$ is given in **Appendix Table VIII**. (Both two- and one-sided tests are possible). Note that α is the joint significance level associated with all $a-1$ tests.

❑ **Example 4:** Use the data in example 1, to conduct comparing mean of type 3 as a control to mean of each type 1 and type 2

a) Manually **b) R program**

The Answer

a) Manually

From Appendix Table VIII: $d_{\alpha}(a-1,f) = d_{0.05}(2,12) = 2.5$

$$n_1 = n_2 = n_3 = 5, \ \sqrt{MSE\left(\frac{1}{n_i}+\frac{1}{n_3}\right)} = \sqrt{16.9\left(\frac{1}{5}+\frac{1}{5}\right)} = 2.6$$

$$D_i = 2.5 \times 2.6 = 6.5$$

$\bar{y}_{i\cdot}$
10.8
22.2
8.4

$\bar{y}_{i\cdot}$	$\bar{y}_{3\cdot}$	$\lvert\bar{y}_{i\cdot}-\bar{y}_{j\cdot}\rvert$	D_i	H_0	Decision
10.8	8.4	2.4	6.5	$H_0: \mu_1 = \mu_3$	H_0 is not rejected
22.2	8.4	13.8	6.5	$H_0: \mu_2 = \mu_3$	H_0 is rejected

b) R program

```
y <- c(9, 12, 10, 8, 15, 20, 21, 23, 17, 30, 6, 5, 8, 16, 7)
          Type <- factor(rep(c("Type1", "Type2", "Type3"), each = 5))
          data1 <- data.frame(y,Type)
# Run the ANOVA model using aov()
          Model <- aov(y ~ Type, data = data1); summary(Model)
#Dunnett test
          # Make Type3 control
          data1$Type <- relevel(data1$Type, ref = "Type3")
          library(multcomp)
          Result_MC <- glht(Model, mcp(Type = "Dunnett")); summary(Result_MC )
```

```
> summary(Result_MC )

         Simultaneous Tests for General Linear Hypotheses

Multiple Comparisons of Means: Dunnett Contrasts


Fit: aov(formula = y ~ Type, data = data1)

Linear Hypotheses:
                   Estimate Std. Error t value Pr(>|t|)
Type1 - Type3 == 0      2.4        2.6   0.923 0.569159
Type2 - Type3 == 0     13.8        2.6   5.308 0.000354 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)
```