3.5.2 Comparisons Among Treatment Means

When the null hypothesis H_0: $\mu_1 = \mu_2 = \cdots = \mu_a$ in the fixed effects model is rejected, it indicates that there are differences between the treatment means, but exactly which means differ is not specified. In this situation, comparisons and analysis among groups of treatment means may be useful. Comparisons between treatment means $\mu_1, \mu_2, \ldots, \mu_a$ are made according to the objective of comparison. There are tow cases for comparisons as follows:

3.5.4 Contrasts

Many multiple comparison methods use the idea of a contrast. This method allows us to compare two groups of treatment means. The steps summarized as follows

- **The Form of a contrast .**

$$\Gamma = c_1\mu_1 + c_2\mu_2 + \cdots c_a\mu_a = \sum_{i=1}^{a} c_i\mu_i = 0$$

where the contrast constants $c_1, c_2, \cdots c_a$ sum to zero, $\sum_{i=1}^{a} c_i = 0$.

- **The point estimate of Γ :**

$$\hat{\Gamma} = C = \sum_{i=1}^{a} c_i\bar{y}_{i.}$$

where C is unbiased estimate to Γ

- **The variance of point estimate C**

$$Var(C) = \frac{\sigma^2}{n}\sum_{i=1}^{3} c_i^2$$

in the case of balance design, $n_1 = n_2 = \cdots = n_a = n$

$$Var(C) = \sigma^2 \sum_{i=1}^{3}\left(\frac{c_i^2}{n_i}\right)$$

in case of unbalance design (some value of n_i) not equal.

- **The unbiased estimation to $Var(C)$ is**

$$\widehat{Var(C)} = \frac{MSE}{n}\sum_{i=1}^{3} c_i^2$$ **balance ,**

$$\widehat{Var(C)} = MSE \sum_{i=1}^{3}\left(\frac{c_i^2}{n_i}\right)$$ **unbalance**

- **The test steps of the contrast by using t test**

- **Null hypothesis:** $H_0: \Gamma = 0$
- **Alternative hypothesis:** $H_1: \Gamma \neq 0$
- **Test statistics:** $T_0 = \frac{\textbf{point estimate to } \Gamma}{S.E_{\textbf{point estimate}}} = \frac{C}{\sqrt{\widehat{Var(C)}}}$
- **Critical value:** $T_c = T_{\left(1-\frac{\alpha}{2}, N-a\right)}$
- **Decision : If** $T_0 > T_c$**, The null hypothesis can be rejected**

❑ Example 1 (3.22 d page 135 in textbook)

The response time in milliseconds was determined for three different types of circuits that could be used in an automatic valve shutoff mechanism. The results from a completely randomized experiment are shown in the following table.

Circuit Type	Response Time				
1	9	12	10	8	15
2	20	21	23	17	30
3	6	5	8	16	7

d) Construct a set of orthogonal contrasts, assuming that at the outset of the experiment you suspected the response time of circuit type 2 to be different from the other two.

The Answer of Example 1

The contrast is: $\mu_2 = \frac{\mu_1+\mu_3}{2} \rightarrow \Gamma = -\mu_1 + 2\mu_2 - \mu_3 = 0, \quad c_i = (-1), (+2), (-1)$

Circuite Type	response Time					$y_{i\cdot}$	$\bar{y}_{i\cdot}$	c_i
1	9	12	10	8	15	54	10.8	-1
2	20	21	23	17	30	111	22.2	+2
3	6	5	8	16	7	42	8.4	-1

- **Create ANOVA table**

S.O.V	*SS*	*df*	*MS*	*F*	*F crit*	*P-value*
Circuite Type	543.6	2	271.8	16.0828	3.8853	0.0004
Error	202.8	12	16.9			
Total	746.4	14				

- **Hypothesis H_0, H_1**

  $H_0: \Gamma = 0 \quad vs \; H_1: \Gamma \neq 0$

- **Compute test statistics**

$$C = \sum_{i=1}^{3} c_i \bar{y}_{i\cdot} = (-1)\bar{y}_{1\cdot} + (2)\bar{y}_{2\cdot} + (-1)\bar{y}_3 = (-1)(10.8) + (2)(22.2) + (-1)(8.4) = 25.2.$$

$$\widehat{Var(C)} = \frac{MSE}{n} \sum_{i=1}^{3} c_i^2 = \frac{16.9}{5}(6) = 20.28$$

$$\mathrm{T}_0 = \frac{\text{point estimate to } \Gamma}{\text{S.E}_{\text{point estimate}}} = \frac{C}{\sqrt{\widehat{Var(C)}}} = \frac{25.2}{\sqrt{20.28}} = 5.596$$

- **Critical value:** $T_c = T_{\left(1-\frac{\alpha}{2}, N-a\right)} = T_{(0.975,12)} = 2.179$

- **Decision :** $T_0 = 5.596 > T_c = 2.179$, **The null hypothesis** $H_0: \Gamma = 0$ **can be rejected, and the response time of circuit type 2 be different from the other two.**

Example 2: From data in exampl1, test the contrast related to comparing circuit type 1 with circuit type 3

- **Hypothesis H_0, H_1**

$H_0: \Gamma = \mu_1 - \mu_3 = 0 \;\; vs \; H_1: \Gamma = \mu_1 - \mu_3 \neq 0$

Contrasts coefficients are $c_i = (+1), (0), (-1)$

Circuite Type	$y_{i.}$	$\bar{y}_{i.}$	c_i
1	54	10.8	1
2	111	22.2	0
3	42	8.4	-1

Compute test statistics

$$C = \sum_{i=1}^{3} c_i \bar{y}_{i.} = (+1)\bar{y}_{1.} + (0)\bar{y}_{2.} + (-1)\bar{y}_3 = (+1)(10.8) + (0)(22.2) + (-1)(8.4) = 2.4$$

$$\widehat{Var(C)} = \frac{MSE}{n} \sum_{i=1}^{3} c_i^2 = \frac{16.9}{5}(2) = 6.76, \qquad \mathrm{T}_0 = \frac{C}{\sqrt{\widehat{Var(C)}}} = \frac{2.4}{\sqrt{6.76}} = 0.923$$

Critical value: $T_c = T_{\left(1-\frac{\alpha}{2}, N-a\right)} = T_{(0.975,12)} = 2.179$

- **Decision :** $T_0 = 0.923 < T_c = 2.179$**, The null hypothesis** H_0: $\Gamma = 0$ **cannot be rejected, and the response time of circuit type 1 does not different from the circuit type 3.**

3.5.5 Orthogonal Contrasts:

a) **The contrasts are named orthogonal contrasts if the following conditions achieved**

1- $\sum_{i=1}^{a} c_i = 0$ **2-** $\sum_{i=1}^{a} c_{il} c_{ik} = 0$

In example 1 and example 2

Circuite Type	c_{i1}	c_{i2}	$c_{i1}c_{i1}$
1	-1	1	-1
2	2	0	0
3	-1	-1	1
Total	0	0	0

b) Sum squares contrast (SS_c) is

Balance case: $$SS_C = \frac{nC^2}{\sum_{i=1}^{a} c_i^2} = \frac{n(\sum_{i=1}^{a} c_i \bar{y}_{i\cdot})^2}{\sum_{i=1}^{a} c_i^2} \quad \text{with } df = 1$$

Unbalance case: $$SS_C = \frac{C^2}{\sum_{i=1}^{a}\left(\frac{c_i^2}{n_i}\right)} = \frac{(\sum_{i=1}^{3} c_i \bar{y}_{i\cdot})^2}{\sum_{i=1}^{a}\left(\frac{c_i^2}{n_i}\right)} \quad \text{with } df = 1$$

c) Under null hypothesis $H_0\colon \Gamma = 0$ **, the test statistics** $F_o = \frac{SS_C}{MSE} \sim F_{(1,N-a)}$

Notes:

d) The number of orthogonal contrasts = $(a-1)$, then analysis of variance table can be expansion as follows

S.O.V	*SS*	*df*	*MS*	*F*	*F crit*
Contrast 1	SS_{c_1}	1	MS_{c_1}	F_{oc_1}	
Contrast 2	SS_2	1	MS_2	F_{oc_2}	
…	…	…	…		
Contrast $(a-1)$	$SS_{c(a-1)}$	1	$MS_{c(a-1)}$	$F_{oc(a-1)}$	
Error	SSE	$N-a$	MSE		
Total	SST_o	$N-1$			

For example, Analysis of variance related to Example 1 and example 2

S.O.V	*SS*	*df*	*MS*	*F*	*F crit*
Contrast 1	529.2	1	529.2	31.31	4.747
Contrast 2	14.4	1	14.4	0.852	4.474
Error	202.8	12	16.9		
Total	746.4	14			

$$SS_{c_1} = \frac{nC^2}{\sum_{i=1}^{a} c_i^2} = \frac{5(25.2)^2}{6} = 529.2$$

$$SS_{c_2} = \frac{nC^2}{\sum_{i=1}^{a} c_i^2} = \frac{5(2.4)^2}{2} = 14.4$$

- Form null and alternative hypotheses in terms of contrasts:

$$H_0: \sum_{i=1}^{a} c_i\mu_i = 0 \quad \textbf{vs} \quad H_1: \sum_{i=1}^{a} c_i\mu_i = 0$$

R program for two contrasts

```
# Enter data
        y <- c(9, 12, 10, 8, 15, 20, 21, 23, 17, 30, 6, 5, 8, 16, 7)
        Type <- factor(rep(c("Type1", "Type2", "Type3"), each = 5))
        data1 <- data.frame(y,Type)
        install.packages("multcomp")
        library(multcomp)

# Run the ANOVA model using aov()
        Fit <- aov(y ~ Type, data = data1)
        summary(Fit)

#The glht() (generalized linear hypothesis tests) function in the multcomp package
# contrasts: Compare Typ2 vs (Typ1+Type3)/2 and Type1 vs Type3
        Cont <- glht(Fit, linfct = mcp(Type = c("-1*Type1 +2* Type2-1*Type3 = 0",
                    "Type1 - Type3 = 0")))
  summary(Cont)
```

Output R program

```
> summary(Fit)
            Df Sum Sq Mean Sq F value   Pr(>F)
Type         2  543.6   271.8   16.08 0.000402 ***
Residuals   12  202.8    16.9
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
> summary(Cont)

         Simultaneous Tests for General Linear Hypotheses

Multiple Comparisons of Means: User-defined Contrasts


Fit: aov(formula = y ~ Type, data = data1)

Linear Hypotheses:
                                         Estimate Std. Error t value Pr(>|t|)
-1 * Type1 + 2 * Type2 - 1 * Type3 == 0   25.200      4.503   5.596 0.000232 ***
Type1 - Type3 == 0                         2.400      2.600   0.923 0.598818
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)
```