# CHAPTER 2

# TREES

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## 1 Definitions- Examples

### 1.1 Definitions:

1. A graph $G$ is *acyclic or forest* if it contains no cycle.
2. A graph $G$ is a *tree* if $G$ is connected and acyclic.

**Remark:**
A graph is acyclic if and only if each of its connected components is a tree.

### 1.2 Examples:

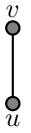
The trees on at most 5 vertices are, up to isomorphy, the following 8 graphs (and the null graph).

1. Tree with one vertex $T = (\{v\}, \emptyset)$



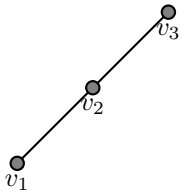

$T$

2. Tree with 2 vertices $T = (\{u, v\}, \{\{u, v\}\})$




$T$

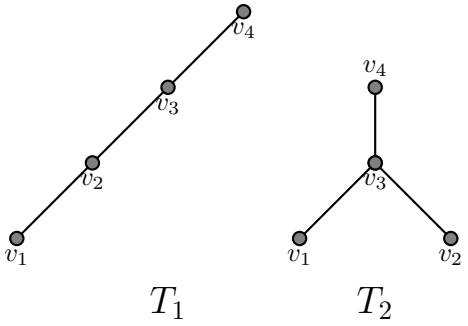
3. Tree with 3 vertices $T = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\})$




$T$

4. Tree with 4 vertices

   $T_1 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\})$,
   and $T_2 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\})$.

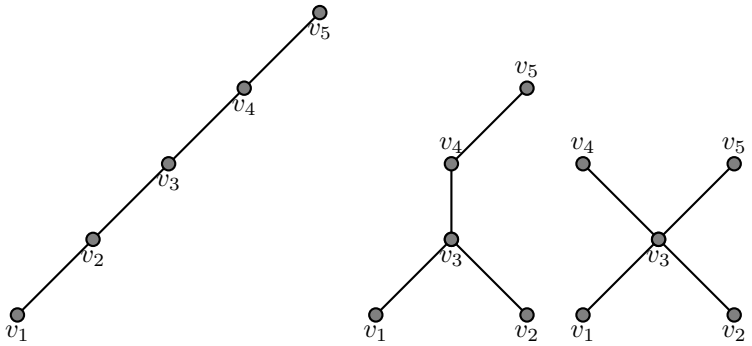



5. Tree with 5 vertices

   $T_1 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\})$,

   $T_2 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\})$,
   and

   $T_3 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_3\}, \{v_5, v_3\}\})$.




# 2 Properties of Trees

## 2.1 Remarks:

Given a graph $G = (V, E)$ and a set $S$ of edges of $G$, we can consider the subgraph of $G$ whose edge set in $S$ and whose vertex set is the set of all ends of edges of $S$. This subgraph is denoted $G[S]$, and called " the edge-induced subgraph $G[S]$".

**Note that:** an edge-induced subgraphs is simply a subgraph without isolated vertices.

**Lemma 2.1**

*Let $G = (V, E)$ be a graph and $u \neq v$ two vertices.*
*If $G$ has two distinct paths $P_1$ and $P_2$ from $u$ to $v$, then $G$ contains a cycle.*

**Proof.**

Consider the edge-induced subgraph $H = G[E(P_1) \cup E(P_2)]$. Each vertex of $H$ is of degree $\geq 2$, then $H$ contains a cycle (So $G$ also).

**Proposition 2.2**

*In a tree, any two distinct vertices are connected by exactly one path.*

**Proof.**

Let $T = (V, E)$ be a tree and $u \neq v \in V$. As $T$ is connected, then $T$ has at least one path $P$ from $u$ to $v$. If $T$ has a path $P' \neq P$, from $u$ to $v$, then by Lemma 2.1, $T$ contains a cycle; contradiction.

**Remarks 2.3**

1. *As any graph in which all degrees are at least two contains a cycle, then every tree has a vertex of degree at most one.*
2. *Thus, any tree $T$ on $n \geq 2$ vertices has at least a vertex of degree 1. Each such a vertex is called "a leaf of $T$".*
3. *Given a graph $G = (V, E)$ and a vertex $x$ of $G$ such that: $d_G(x) = 1$. Then:*
   (a) *($G$ is connected) if and only if ($G - x$ is connected).*
   (b) *($G$ is acyclic) if and only if ($G - x$ is acyclic).*
4. **Note** *Thus, given a tree $T = (V, E)$ on $n \geq 2$ vertices, then: for each leaf $x$ of $T$, $T - x$ is a tree.*

**Lemma 2.4**

*Given an acyclic graph $G = (V, E)$ with at least one edge (i.e nonempty acyclic graph), then $G$ has at least two vertices of degree 1.*

**Proof.**

Consider a path $P = (u_1, ..., u_p)$ of $G$, with maximum length (in particular: $p \geq 2$). Then, $d(u_1) \geq 1$ and $d(u_p) \geq 1$.
**If** $d(u_1) > 1$ **(resp.** $d(u_p) > 1$**)**, then there is $y \in N(u_1) \setminus \{u_2\}$ (resp. $y \in N(u_p) \setminus \{u_{p-1}\}$).
We distinguish two cases as follows:
**Firstly:** If $y = u_i$, for some $i \in \{1, ..., p\}$, then $p \geq 3$ and $C = (u_1, u_2, ..., u_i = y, u_1)$ is a cycle of $G$ (resp. $C = (u_i = y, u_{i+1}, ..., u_p, u_i = y)$ is a cycle of $G$); contradiction.
**Secondly:** If $y \notin \{u_j; 1 \leq j \leq p\}$, then $P' = (y, u_1, ..., u_p)$ is a path of $G$ and its length $l$ verifies: $l >$ length of $P$; contradiction.
Thus, $d_G(u_1) = d_G(u_p) = 1$.

**Proposition 2.5**

*Every tree on $n \geq 2$ vertices, has at least two leaves.*

**Proof.**

As a tree $T$ on $n \geq 2$ vertices is an acyclic graph with at least one edges, we conclude by Lemma 2.4

**Proposition 2.6**

*For every tree $T$ on $n \geq 1$ vertices, we have: $e(T) = n - 1$ (i.e: $e(T) = v(V) - 1$).*

**Proof.**

**By induction on** $n$**.** If $n = 1$, then $T \simeq K_1$ and then $e(T) = 0 = v(V) - 1$.
Let $n \geq 1$ be an integer and assume that for every tree $T$ on $n \geq 1$ vertices, $e(T) = n-1$. Consider a tree $T' = (V, E)$ with: $v(T') = |V| = n + 1$. By Proposition 2.5, we may consider a leaf $u$ of $T'$. As $T' - u$ is a tree on $n$ vertices, then by hypothesis (of induction); $e(T' - u) = v(T' - u) - 1$. But, $e(T' - u) = e(T') - 1$ (because $d_{T'}(u) = 1$) and $v(T' - u) = |V(T') \setminus \{u\}| = v(T') - 1 = n$. So, $e(T') - 1 = n - 1$. Thus, $e(T') = n = v(T') - 1$.

# 3 Characterization of Trees

**Lemma 3.1**

*Let $G = (V, E)$ be a graph without isolated vertex such that: $|V| = n \geq 2$ and $|E| = n - 1$. Then $G$ has at least two vertices of degree 1.*

**Proof.**

Let $A = \{u \in V : d(u) = 1\}$ and $B = V \setminus A$. $2|E| = \sum_{u \in V} d(u) = \sum_{u \in A} d(u) + \sum_{u \in B} d(u)$.
So, $2(n-1) = |A| + \sum_{u \in B} d(u)$. As: $\forall u \in B,\ d(u) \geq 2$, then: $2(n-1) \geq |A| + 2|B| = |A| + 2(n - |A|)$.
Thus, $2n - 2 \geq -|A| + 2n$ and then: $|A| \geq 2$.

**Theorem 3.2**

*Let $G = (V, E)$ be a graph with: $|V| = n \geq 1$. The following assertions are equivalent.*

1. *$G$ is a tree.*
2. *$G$ is connected and $|E| = n - 1$.*
3. *$G$ is acyclic and $|E| = n - 1$.*

**Proof.**

- "1. $\Rightarrow$ 2." If $G$ is a tree, then $G$ is connected (by definition) and $|E| = n - 1$ by Proposition 2.6.
- "2. $\Rightarrow$ 3." By induction on $n$ ( for "$G$ is acyclic").
  $\rightarrow$ For $n = 1$ or $n = 2$, the result is immediate.
  $\rightarrow$ For $n \geq 2$ such that the result is true for $n$. Consider a graph $G' = (V', E')$ such that $G'$ is connected, $|V'| = n + 1$ and $|E'| = n$. By Lemma 3.1, consider a vertex $u$ of $G'$ such that: $d_{G'}(u) = 1$ (indeed there are at least two such vertices).
  Clearly, $G' - u$ is connected with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = |E'| - 1 = n - 1$.
  Thus, by hypothesis (of induction), $(G' - u)$ is acyclic. As, $d_{G'}(u) = 1$, then $G'$ is also acyclic.
- "3. $\Rightarrow$ 1." By induction on $n$.
  $\rightarrow$ For $n = 1$, the result is trivial.
  $\rightarrow$ Assume that the result (i.e. this implication) is true for some $n \geq 1$ and consider a graph $G' = (V', E')$ such that $G'$ is acyclic, $|V'| = n + 1$ and $|E'| = n$. By Lemma 2.4, consider a vertex $u$ of $G'$ such that: $d_{G'}(u) = 1$.
  Thus, $G' - u$ is (also) acyclic with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = n - 1$.
  Thus, by hypothesis (of induction), $(G' - u)$ is a tree, then is connected. Thus, $G'$ is also connected (and then $G'$ is a tree, because $G'$ is acyclic by hypothesis).

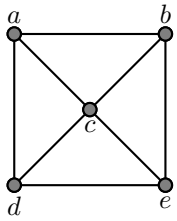
# 4 Spanning trees

**Definition 4.1**

1. *Given a graph $G = (V, E)$, a subgraph of $G$ which is a tree is called "a subtree of $G$".*
2. *A spanning subgraph of $G$ which is a tree is called spanning tree of $G$.*

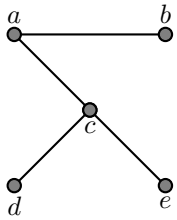
**Example 4.2**

*Let $G = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{e, d\}, \{e, c\}, \{d, c\}\})$ be a graph.*



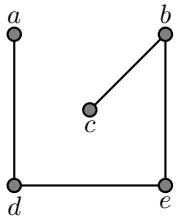

$G$

*Then $T_1 = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{e, c\}, \{d, c\}\})$ is spanning tree of $G$.*




$T_1$

*Then $T_2 = (\{a, b, c, d, e\}, \{\{a, d\}, \{e, d\}, \{b, e\}, \{b, c\}\})$ is spanning tree of $G$.*




$T_2$

**Remarks 4.3**

1. *Recall that a result in chapter 2: Given an edge $e$ of a graph $G$,*
   *($e$ is a bridge of $G$) if and only if ( $e$ does not lie on any cycle of $G$).*
2. *A bridge of $G$ which is also called " a cut edge of $G$" (See Book of "Bondy and Murty").*

**Theorem 4.4**

*A graph is connected if and only if it has a spanning tree.*

**Proof.**

- "⇐" If a graph $G$ has a spanning tree $T$, then two distinct vertices of $G$ (and then of $T$) are connected by a path in $T$, which is also a path in $G$; so $G$ is connected.
- "⇒" Consider a connected graph $G = (V, E)$ which is not tree. Given an edge $e$ of a cycle of $G$, $G - e = (V, E \setminus \{e\})$ is a spanning subgraph of $G$ which is also connected (because by remark $e$ is not a bridge of $G$). By repeating this process of deleting edges in cycles until every edge which remains does not lie in a cycle and then it is a cut edge (i.e bridge), we obtain a spanning subgraph $T$ of $G$ such that: $T$ is connected and acyclic. So, $T$ is a spanning tree of $G$.

**Corollary 4.5**
*If $G = (V, E)$ is a connected graph with $|V| = n \geq 1$, then $|E| \geq n - 1$.*

**Proof.**
By Theorem 4.4, $G$ has a spanning tree $T$. By Theorem 3.2, $|E(T)| = n - 1$. As, $E(T) \subset E(G) = E$, then: $|E| \geq n - 1$.

# 5 Characterization of bipartite Graphs

**Remarks 5.1**

1. *A graph $G$ is a bipartite if and only if each of its connected components is bipartite.*
2. *A graph $G$ contains an odd cycle if and only if one of its connected components contains an odd cycle.*

**Theorem 5.2**
*A graph is bipartite if and only if it contains no odd cycles.*

**Proof.**
By remark 5.1, it suffices to prove the theorem in the case of connected graphs.

- "⇒" Let $G = G[X, Y]$ be a connected bipartite graph. The vertices of any path in $G$ belong alternately to $X$ and to $Y$. Thus, all paths connecting 2 vertices in different parts (one in $X$ and the other in $Y$) is of odd length, and all paths connecting 2 vertices in the same part ($X$ or $Y$) are of even length.
  It follows that every cycle $(\alpha_1, \alpha_2, ..., \alpha_p, \alpha_1)$ in $G$ is of even length (because $\alpha_1$ and $\alpha_p$ are in different parts, so $l(\alpha_1, \alpha_2, ..., \alpha_p)$ is odd and then $l(\alpha_1, \alpha_2, ..., \alpha_p, \alpha_1)$ is even).
- "⇐" Conversely; consider a connected graph $G$ without odd cycles: By Theorem 4.4, $G$ has a spanning tree $T = (V, E_1)$ (where $G = (V, E)$). We may assume that: $|V| \geq 2$.
  Let $x \in V$. By Proposition 2.2, for each $v \in V \setminus \{x\}$, there is only one path $P_v$, in $T$, connecting $v$ to $x$.
  Let: $X = \{x\} \cup \{v \in V \setminus \{x\} : l(P_v) \text{ is even}\}$ and $Y = V \setminus X$.
  Note That: $Y = \{v \in V \setminus \{x\} : l(P_v) \text{ is odd}\}$, and $Y \neq \emptyset$ (because $N_T(x) \neq \emptyset$ and $N_T(x) \subseteq Y$).

Thus, $\{X, Y\}$ is a partition of $V$ (because $X \neq \emptyset$, $Y \neq \emptyset$, and $X \cap Y = \emptyset$).

**Fact 1:** $T[X]$ is an empty graph.

**Indeed:**
$\rightarrow \forall \alpha \in X,\ \{\alpha, x\} \notin E(T)$ because $N_T(x) \subseteq Y$.
$\rightarrow$ Assume that $T[X]$ is a nonempty graph. So, there is an edge $\{\alpha, \beta\} \subseteq X \setminus \{x\}$.
Assume, for example, that $l(P_\alpha) \leq l(P_\beta)$ and let $P_\alpha = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha)$. As, $l(P_\alpha) \leq l(P_\beta)$, then $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So, $P'_\alpha = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha, \beta)$ is a path in $T$ connecting $x$ to $\beta$. It ensues, that: $l(P_\beta) = l(P'_\alpha) = l(P_\alpha) + 1$. Then $l(P_\beta)$ is odd (because $l(P_\alpha)$ is even; $(\alpha \in X)$); contradicts the fact that $\beta \in X$.

**Fact 2:** $T[Y]$ is an empty graph.

**Indeed:**
Assume that $T([Y]$ is a nonempty graph and consider an edge $\{\alpha, \beta\}$ of $T([Y]$ with: $l(P_\alpha) \leq l(P_\beta)$. Denote: $P_\alpha = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha)$. Then, $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So, $P_\beta = (\alpha_1 = x, \alpha_2, ..., \alpha_p = \alpha, \beta)$ is a path in $T$ connecting $x$ to $\beta$. So, $l(P_\beta) = l(P_\alpha) + 1$. Contradiction.

$\rightarrow$ Thus, $T$ is bipartite graph with the partition $\{X, Y\}$.

**Now**, we will prove that $G$ is bipartite with $\{X, Y\}$ as a bipartition.
For this, consider an edge $e \in E \setminus E_1 = E(G) \setminus E(T)$. Denote: $e = \{u, v\}$.
Let $P = (\alpha_1 = u, \alpha_2, ..., \alpha_q = v)$ the unique path in $T$, from $u$ to $v$. By hypothesis, the cycle $C = P + e$ is of even length. Thus, $l(P) = l(C) - 1$ is odd. As $P$ is a path of the bipartite graph $T = T[X, Y]$, then $u$ and $v$ belong to distinct parts (i.e one is in $X$ and the other is in $Y$).
Thus, every edge $e'$ of $G$ is such that: one end of $e'$ is in $X$ and the other end is in $Y$.
So $G$ is bipartite and $G = G[X, Y]$.

**Corollary 5.3**
*Every tree is a bipartite graph.*

**Proof.**
As a tree is without cycles, this corollary is an immediate consequence of Theorem 5.2.

# 6 Exercises of Trees

## N.B: All the graphs considered here are nontrivial

**Exercise 6.1**

*Let $F$ be a forest of order $p$ and size $q$ having $k$ connected components. Obtain an expression for $q$ in terms of $p$ and $k$.*

**Exercise 6.2**

1. *(a) Find all the trees, up to isomorphism, $T$ is a regular graph.*
   *(b) Find all the trees, up to isomorphism, $T$ is a complete bipartite graph $K_{p,q}$ (where $p, q \geq 1$).*
2. *(a) Find all the trees $T$, up to isomorphism, on $\{1, 2, ..., n\}$ where $n \geq 3$, and $T$ has exactly $n-1$ leaves.*
   *(b) Find all the trees $T$, up to isomorphism, on $\{1, 2, ..., n\}$ where $n \geq 3$, and $T$ has exactly 2 leaves.*

**Exercise 6.3** *Given an integer $n \geq 2$, show that a nondecreasing sequence $D = (d_1, d_2, \ldots, d_n)$ of positive integers is the degree sequence of some tree if and only if*

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

Hint: *Prove that if $n \geq 3$ and the sum is $2n-2$, then $d_1 = 1$ and $d_n > 1$, and use induction on $n$.*

**Solution.** Suppose first that $D$ is the degree sequence of a tree $T$ on $n$ vertices. A tree on $n$ vertices has $n-1$ edges, so the Handshaking Lemma gives

$$\sum_{i=1}^{n} d_i = 2|E(T)| = 2(n-1).$$

Conversely, suppose that $D$ is a nondecreasing sequence of positive integers with sum $2n-2$. We prove the assertion by induction on $n$.

**Base case:** $n = 2$**.** Since $d_1, d_2 \geq 1$ and $d_1 + d_2 = 2$, we have $D = (1, 1)$, which is the degree sequence of $K_2$.

**Inductive step.** Let $n \geq 3$ and assume the assertion holds for sequences of $n-1$ positive integers. If $d_1 \geq 2$, then every term is at least 2, so $\sum_{i=1}^{n} d_i \geq 2n$, a contradiction. Hence $d_1 = 1$. If $d_n = 1$, then every term equals 1, so the sum is $n < 2n - 2$, again a contradiction. Thus $d_n > 1$.

Remove $d_1 = 1$ and decrease $d_n$ by 1. The resulting sequence

$$D' = (d_2, \ldots, d_{n-1}, d_n - 1)$$

contains $n-1$ positive integers and has sum

$$\sum_{i=2}^{n-1} d_i + (d_n - 1) = 2n - 4 = 2(n-1) - 2.$$

After rearranging $D'$ in nondecreasing order if necessary, the induction hypothesis yields a tree $T'$ realizing $D'$. Choose a vertex $u$ of $T'$ with degree $d_n - 1$. Add a new vertex $v$ and the single edge $uv$. The resulting graph $T$ is connected; it has no cycle, since $T'$ has no cycle and the new vertex has degree 1. Therefore $T$ is a tree.

The new vertex has degree $1 = d_1$, the degree of $u$ becomes $d_n$, and all other degrees remain unchanged. Hence the degree sequence of $T$, arranged in nondecreasing order, is $D$. □

**Exercise 6.4**

*Show that any tree $T$ has at least $\Delta(T)$ leaves.*

**Exercise 6.5**

*Let $T$ be a tree with $v(T) = n \geq 3$, and denote $x_i = |\{v \in V(T) : d_T(v) = i\}|$.*

1. *Show that* $\sum_{i=3}^{n-1}(i-2)x_i = x_1 - 2$,

2. *Count the number of nonisomorphic trees that have 5 leaves and no vertices of degree 2.*

**Exercise 6.6**

*Let $G$ be a connected graph.*

1. *Let $T$ be a spanning tree of $G$. Show that every cycle of $G$ has an edge that is in the complement $\overline{T}$.*

2. *Let $e \in E(G)$. Show that $e$ is a bridge of $G$ if and only if $e$ belongs to every spanning tree of $G$.*

**Exercise 6.7**

*Let $T$ be a tree of order $k$. Show that any graph $G$ with $\delta(G) \geq k-1$ has a subgraph isomorphic to $T$.*

*(Hint: Use an induction on $k$).*

**Exercise 6.8**

*Let $T$ be a tree of order $n \geq 2$.*

1. *Let $u \in V(T)$, and suppose that $P : (v_1 = u, v_2, ..., v_p = v)$ is a maximal path in $T$ such that $u$ is an end of $P$. Show that $v$ is a leaf of $T$.*

2. *Show that there is a vertex $v$ of $T$ such that $N_T(v)$ consists of leaves excepting possibly one neighbour.*

**Exercise 6.9**

*Let $A \subseteq \mathbb{N}$ such that $0 \notin A$ and $1 \in A$. Show that there is a tree $T$ such that $A = \{d_T(v) : v \in V(T)\}$.*
*(Hint: Use an induction on $n = |A|$).*