

CHAPTER 2

TREES

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1 Definitions- Examples

1.1 Definitions:

1. A graph G is *acyclic or forest* if it contains no cycle.
2. A graph G is a *tree* if G is connected and acyclic.

Remark:

A graph is acyclic if and only if each of its connected components is a tree.

1.2 Examples:

The trees on at most 5 vertices are, up to isomorphism, the following 8 graphs (and the null graph).

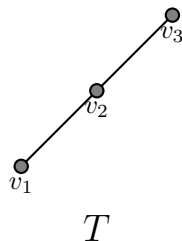
1. Tree with one vertex $T = (\{v\}, \emptyset)$



2. Tree with 2 vertices $T = (\{u, v\}, \{\{u, v\}\})$



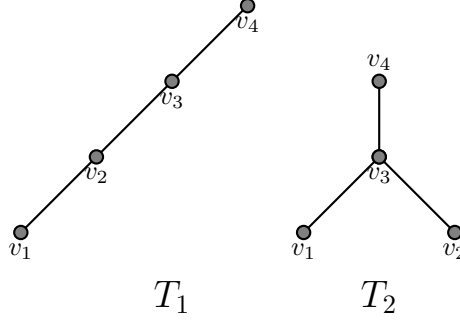
3. Tree with 3 vertices $T = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\})$



4. Tree with 4 vertices

$$T_1 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}),$$

$$\text{and } T_2 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\}).$$



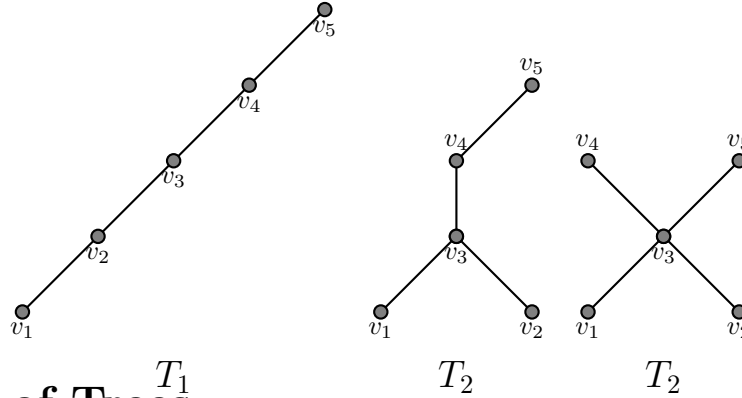
5. Tree with 5 vertices

$$T_1 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}),$$

$$T_2 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}),$$

and

$$T_3 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_3\}, \{v_5, v_3\}\}).$$



2 Properties of Trees

2.1 Remarks:

Given a graph $G = (V, E)$ and a set S of edges of G , we can consider the subgraph of G whose edge set is S and whose vertex set is the set of all ends of edges of S . This subgraph is denoted $G[S]$, and called "the edge-induced subgraph $G[S]$ ".

Note that: an edge-induced subgraph is simply a subgraph without isolated vertices.

Lemma 2.1

Let $G = (V, E)$ be a graph and $u \neq v$ two vertices.

If G has two distinct paths P_1 and P_2 from u to v , then G contains a cycle.

Proof.

Consider the edge-induced subgraph $H = G[E(P_1) \cup E(P_2)]$. Each vertex of H is of degree ≥ 2 , then H contains a cycle (So G also).

Proposition 2.2

In a tree, any two distinct vertices are connected by exactly one path.

Proof.

Let $T = (V, E)$ be a tree and $u \neq v \in V$. As T is connected, then T has at least one path P from u to v . If T has a path $P' \neq P$, from u to v , then by Lemma 2.1, T contains a cycle; contradiction.

Remarks 2.3

1. *As any graph in which all degrees are at least two contains a cycle, then every tree has a vertex of degree at most one.*
2. *Thus, any tree T on $n \geq 2$ vertices has at least a vertex of degree 1. Each such a vertex is called "a leaf of T ".*
3. *Given a graph $G = (V, E)$ and a vertex x of G such that: $d_G(x) = 1$. Then:*
 - (a) *(G is connected) if and only if ($G - x$ is connected).*
 - (b) *(G is acyclic) if and only if ($G - x$ is acyclic).*
4. **Note** *Thus, given a tree $T = (V, E)$ on $n \geq 2$ vertices, then: for each leaf x of T , $T - x$ is a tree.*

Lemma 2.4

Given an acyclic graph $G = (V, E)$ with at least one edge (i.e nonempty acyclic graph), then G has at least two vertices of degree 1.

Proof.

Consider a path $P = (u_1, \dots, u_p)$ of G , with maximum length (in particular: $p \geq 2$). Then, $d(u_1) \geq 1$ and $d(u_p) \geq 1$.

If $d(u_1) > 1$ (resp. $d(u_p) > 1$), then there is $y \in N(u_1) \setminus \{u_2\}$ (resp. $y \in N(u_p) \setminus \{u_{p-1}\}$).

We distinguish two cases as follows:

Firstly: If $y = u_i$, for some $i \in \{1, \dots, p\}$, then $p \geq 3$ and $C = (u_1, u_2, \dots, u_i = y, u_1)$ is a cycle of G (resp. $C = (u_i = y, u_{i+1}, \dots, u_p, u_i = y)$ is a cycle of G); contradiction.

Secondly: If $y \notin \{u_j; 1 \leq j \leq p\}$, then $P' = (y, u_1, \dots, u_p)$ is a path of G and its length l verifies: $l > \text{length of } P$; contradiction.

Thus, $d_G(u_1) = d_G(u_p) = 1$.

Proposition 2.5

Every tree on $n \geq 2$ vertices, has at least two leaves.

Proof.

As a tree T on $n \geq 2$ vertices is an acyclic graph with at least one edges, we conclude by Lemma 2.4

Proposition 2.6

For every tree T on $n \geq 1$ vertices, we have: $e(T) = n - 1$ (i.e: $e(T) = v(V) - 1$).

Proof.

By induction on n . If $n = 1$, then $T \simeq K_1$ and then $e(T) = 0 = v(V) - 1$.
Let $n \geq 1$ be an integer and assume that for every tree T on $n \geq 1$ vertices, $e(T) = n - 1$. Consider a tree $T' = (V, E)$ with: $v(T') = |V| = n + 1$. By Proposition 2.5, we may consider a leaf u of T' . As $T' - u$ is a tree on n vertices, then by hypothesis (of induction); $e(T' - u) = v(T' - u) - 1$. But, $e(T' - u) = e(T') - 1$ (because $d_{T'}(u) = 1$) and $v(T' - u) = |V(T') \setminus \{u\}| = v(T') - 1 = n$. So, $e(T') - 1 = n - 1$. Thus, $e(T') = n = v(T') - 1$.

3 Characterization of Trees

Lemma 3.1

Let $G = (V, E)$ be a graph without isolated vertex such that: $|V| = n \geq 2$ and $|E| = n - 1$. Then G has at least two vertices of degree 1.

Proof.

$$\text{Let } A = \{u \in V : d(u) = 1\} \text{ and } B = V \setminus A. \quad 2|E| = \sum_{u \in V} d(u) = \sum_{u \in A} d(u) + \sum_{u \in B} d(u).$$

So, $2(n - 1) = |A| + \sum_{u \in B} d(u)$. As: $\forall u \in B, d(u) \geq 2$, then: $2(n - 1) \geq |A| + 2|B| = |A| + 2(n - |A|)$.

Thus, $2n - 2 \geq -|A| + 2n$ and then: $|A| \geq 2$.

Theorem 3.2

Let $G = (V, E)$ be a graph with: $|V| = n \geq 1$. The following assertions are equivalent.

1. G is a tree.
2. G is connected and $|E| = n - 1$.
3. G is acyclic and $|E| = n - 1$.

Proof.

- "1. $\Rightarrow$ 2." If G is a tree, then G is connected (by definition) and $|E| = n - 1$ by Proposition 2.6.
- "2. $\Rightarrow$ 3." By induction on n (for " G is acyclic").
 $\rightarrow$ For $n = 1$ or $n = 2$, the result is immediate.
 $\rightarrow$ For $n \geq 2$ such that the result is true for n . Consider a graph $G' = (V', E')$ such that G' is connected, $|V'| = n + 1$ and $|E'| = n$. By Lemma 3.1, consider a vertex u of G' such that: $d_{G'}(u) = 1$ (indeed there are at least two such vertices).
 Clearly, $G' - u$ is connected with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = |E'| - 1 = n - 1$.
 Thus, by hypothesis (of induction), $(G' - u)$ is acyclic. As, $d_{G'}(u) = 1$, then G' is also acyclic.
- "3. $\Rightarrow$ 1." By induction on n .
 $\rightarrow$ For $n = 1$, the result is trivial.
 $\rightarrow$ Assume that the result (i.e. this implication) is true for some $n \geq 1$ and consider a graph $G' = (V', E')$ such that G' is acyclic, $|V'| = n + 1$ and $|E'| = n$. By Lemma 2.4, consider a vertex u of G' such that: $d_{G'}(u) = 1$.
 Thus, $G' - u$ is (also) acyclic with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = n - 1$.
 Thus, by hypothesis (of induction), $(G' - u)$ is a tree, then is connected. Thus, G' is also connected (and then G' is a tree, because G' is acyclic by hypothesis).

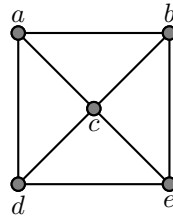
4 Spanning trees

Definition 4.1

1. Given a graph $G = (V, E)$, a subgraph of G which is a tree is called "a subtree of G ".
2. A spanning subgraph of G which is a tree is called spanning tree of G .

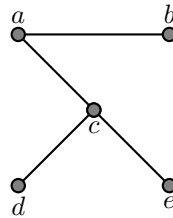
Example 4.2

Let $G = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{e, d\}, \{e, c\}, \{d, c\}\})$ be a graph.



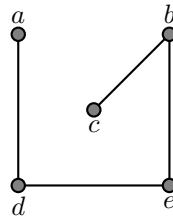
G

Then $T_1 = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{e, c\}, \{d, c\}\})$ is spanning tree of G .



T_1

Then $T_2 = (\{a, b, c, d, e\}, \{\{a, d\}, \{e, d\}, \{b, e\}, \{b, c\}\})$ is spanning tree of G .



T_2

Remarks 4.3

1. Recall that a result in chapter 2: Given an edge e of a graph G , (e is a bridge of G) if and only if (e does not lie on any cycle of G).
2. A bridge of G which is also called "a cut edge of G " (See Book of "Bondy and Murty").

Theorem 4.4

A graph is connected if and only if it has a spanning tree.

Proof.

- " $\Leftarrow$ " If a graph G has a spanning tree T , then two distinct vertices of G (and then of T) are connected by a path in T , which is also a path in G ; so G is connected.
- " $\Rightarrow$ " Consider a connected graph $G = (V, E)$ which is not tree. Given an edge e of a cycle of G , $G - e = (V, E \setminus \{e\})$ is a spanning subgraph of G which is also connected (because by remark e is not a bridge of G). By repeating this process of deleting edges in cycles until every edge which remains does not lie in a cycle and then it is a cut edge (i.e bridge), we obtain a spanning subgraph T of G such that: T is connected and acyclic. So, T is a spanning tree of G .

Corollary 4.5

If $G = (V, E)$ is a connected graph with $|V| = n \geq 1$, then $|E| \geq n - 1$.

Proof.

By Theorem 4.4, G has a spanning tree T . By Theorem 3.2, $|E(T)| = n - 1$. As, $E(T) \subset E(G) = E$, then: $|E| \geq n - 1$.

5 Characterization of bipartite Graphs

Remarks 5.1

1. *A graph G is a bipartite if and only if each of its connected components is bipartite.*
2. *A graph G contains an odd cycle if and only if one of its connected components contains an odd cycle.*

Theorem 5.2

A graph is bipartite if and only if it contains no odd cycles.

Proof.

By remark 5.1, it suffices to prove the theorem in the case of connected graphs.

- " $\Rightarrow$ " Let $G = G[X, Y]$ be a connected bipartite graph. The vertices of any path in G belong alternately to X and to Y . Thus, all paths connecting 2 vertices in different parts (one in X and the other in Y) is of odd length, and all paths connecting 2 vertices in the same part (X or Y) are of even length.
It follows that every cycle $(\alpha_1, \alpha_2, \dots, \alpha_p, \alpha_1)$ in G is of even length (because α_1 and α_p are in different parts, so $l(\alpha_1, \alpha_2, \dots, \alpha_p)$ is odd and then $l(\alpha_1, \alpha_2, \dots, \alpha_p, \alpha_1)$ is even).
- " $\Leftarrow$ " Conversely; consider a connected graph G without odd cycles: By Theorem 4.4, G has a spanning tree $T = (V, E_1)$ (where $G = (V, E)$). We may assume that: $|V| \geq 2$.
Let $x \in V$. By Proposition 2.2, for each $v \in V \setminus \{x\}$, there is only one path P_v , in T , connecting v to x .
Let: $X = \{x\} \cup \{v \in V \setminus \{x\} : l(P_v) \text{ is even}\}$ and $Y = V \setminus X$.
Note That: $Y = \{v \in V \setminus \{x\} : l(P_v) \text{ is odd}\}$, and $Y \neq \emptyset$ (because $N_T(x) \neq \emptyset$ and $N_T(x) \subseteq Y$).

Thus, $\{X, Y\}$ is a partition of V (because $X \neq \emptyset$, $Y \neq \emptyset$, and $X \cap Y = \emptyset$).

Fact 1: $T[X]$ is an empty graph.

Indeed:

$\rightarrow \forall \alpha \in X, \{\alpha, x\} \notin E(T)$ because $N_T(x) \subseteq Y$.

$\rightarrow$ Assume that $T[X]$ is a nonempty graph. So, there is an edge $\{\alpha, \beta\} \subseteq X \setminus \{x\}$.

Assume, for example, that $l(P_\alpha) \leq l(P_\beta)$ and let $P_\alpha = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha)$. As, $l(P_\alpha) \leq l(P_\beta)$, then $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So, $P'_\alpha = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha, \beta)$ is a path in T connecting x to β . It ensues, that: $l(P_\beta) = l(P'_\alpha) = l(P_\alpha) + 1$. Then $l(P_\beta)$ is odd (because $l(P_\alpha)$ is even; $(\alpha \in X)$); contradicts the fact that $\beta \in X$.

Fact 2: $T[Y]$ is an empty graph.

Indeed:

Assume that $T[Y]$ is a nonempty graph and consider an edge $\{\alpha, \beta\}$ of $T[Y]$ with:

$l(P_\alpha) \leq l(P_\beta)$. Denote: $P_\alpha = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha)$. Then, $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So,

$P_\beta = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha, \beta)$ is a path in T connecting x to β . So, $l(P_\beta) = l(P_\alpha) + 1$. Contradiction.

$\rightarrow$ Thus, T is bipartite graph with the partition $\{X, Y\}$.

Now, we will prove that G is bipartite with $\{X, Y\}$ as a bipartition.

For this, consider an edge $e \in E \setminus E_1 = E(G) \setminus E(T)$. Denote: $e = \{u, v\}$.

Let $P = (\alpha_1 = u, \alpha_2, \dots, \alpha_q = v)$ the unique path in T , from u to v . By hypothesis, the cycle $C = P + e$ is of even length. Thus, $l(P) = l(C) - 1$ is odd. As P is a path of the bipartite graph $T = T[X, Y]$, then u and v belong to distinct parts (i.e one is in X and the other is in Y).

Thus, every edge e' of G is such that: one end of e' is in X and the other end is in Y .

So G is bipartite and $G = G[X, Y]$.

Corollary 5.3

Every tree is a bipartite graph.

Proof.

As a tree is without cycles, this corollary is an immediate consequence of Theorem 5.2.

6 Exercises of Trees

N.B: All the graphs considered here are nontrivial

Exercise 6.1

Let F be a forest of order p and size q having k connected components. Obtain an expression for q in terms of p and k .

Exercise 6.2

1. (a) Find all the trees, up to isomorphism, T is a regular graph.
 (b) Find all the trees, up to isomorphism, T is a complete bipartite graph $K_{p,q}$ (where $p, q \geq 1$).
2. (a) Find all the trees T , up to isomorphism, on $\{1, 2, \dots, n\}$ where $n \geq 3$, and T has exactly $n - 1$ leaves.
 (b) Find all the trees T , up to isomorphism, on $\{1, 2, \dots, n\}$ where $n \geq 3$, and T has exactly 2 leaves.

Exercise 6.3 Given an integer $n \geq 2$, show that a nondecreasing sequence $D = (d_1, d_2, \dots, d_n)$ of positive integers is the degree sequence of some tree if and only if

$$\sum_{i=1}^n d_i = 2n - 2.$$

Hint: Prove that if $n \geq 3$ and the sum is $2n - 2$, then $d_1 = 1$ and $d_n > 1$, and use induction on n .

Solution. Suppose first that D is the degree sequence of a tree T on n vertices. A tree on n vertices has $n - 1$ edges, so the Handshaking Lemma gives

$$\sum_{i=1}^n d_i = 2|E(T)| = 2(n - 1).$$

Conversely, suppose that D is a nondecreasing sequence of positive integers with sum $2n - 2$. We prove the assertion by induction on n .

Base case: $n = 2$. Since $d_1, d_2 \geq 1$ and $d_1 + d_2 = 2$, we have $D = (1, 1)$, which is the degree sequence of K_2 .

Inductive step. Let $n \geq 3$ and assume the assertion holds for sequences of $n - 1$ positive integers.

If $d_1 \geq 2$, then every term is at least 2, so $\sum_{i=1}^n d_i \geq 2n$, a contradiction. Hence $d_1 = 1$. If $d_n = 1$, then every term equals 1, so the sum is $n < 2n - 2$, again a contradiction. Thus $d_n > 1$.

Remove $d_1 = 1$ and decrease d_n by 1. The resulting sequence

$$D' = (d_2, \dots, d_{n-1}, d_n - 1)$$

contains $n - 1$ positive integers and has sum

$$\sum_{i=2}^{n-1} d_i + (d_n - 1) = 2n - 4 = 2(n - 1) - 2.$$

After rearranging D' in nondecreasing order if necessary, the induction hypothesis yields a tree T' realizing D' . Choose a vertex u of T' with degree $d_n - 1$. Add a new vertex v and the single edge uv . The resulting graph T is connected; it has no cycle, since T' has no cycle and the new vertex has degree 1. Therefore T is a tree.

The new vertex has degree $1 = d_1$, the degree of u becomes d_n , and all other degrees remain unchanged. Hence the degree sequence of T , arranged in nondecreasing order, is D . $\square$

Exercise 6.4

Show that any tree T has at least $\Delta(T)$ leaves.

Exercise 6.5

Let T be a tree with $v(T) = n \geq 3$, and denote $x_i = |\{v \in V(T) : d_T(v) = i\}|$.

1. Show that $\sum_{i=3}^{n-1} (i-2)x_i = x_1 - 2$,
2. Count the number of nonisomorphic trees that have 5 leaves and no vertices of degree 2.

Exercise 6.6

Let G be a connected graph.

1. Let T be a spanning tree of G . Show that every cycle of G has an edge that is in the complement $\bar{T}$.
2. Let $e \in E(G)$. Show that e is a bridge of G if and only if e belongs to every spanning tree of G .

Exercise 6.7

Let T be a tree of order k . Show that any graph G with $\delta(G) \geq k-1$ has a subgraph isomorphic to T .

(Hint: Use an induction on k).

Exercise 6.8

Let T be a tree of order $n \geq 2$.

1. Let $u \in V(T)$, and suppose that $P : (v_1 = u, v_2, \dots, v_p = v)$ is a maximal path in T such that u is an end of P . Show that v is a leaf of T .
2. Show that there is a vertex v of T such that $N_T(v)$ consists of leaves excepting possibly one neighbour.

Exercise 6.9

Let $A \subseteq \mathbb{N}$ such that $0 \notin A$ and $1 \in A$. Show that there is a tree T such that $A = \{d_T(v) : v \in V(T)\}$.

(Hint: Use an induction on $n = |A|$).

Exercise 6.10

1. Define the Prüfer code $b(T) = (p_1, p_2, \dots, p_{n-2})$ of a tree T of order n .
2. Prove Cayley's Theorem: There are exactly n^{n-2} trees on $\{1, 2, \dots, n\}$.

Solution.

1. Definition of the Prüfer Code $b(T)$

Let T be a labeled tree of order $n \geq 3$ with vertex set $V = \{1, 2, \dots, n\}$. The **Prüfer code** $b(T) = (p_1, p_2, \dots, p_{n-2})$ is a sequence of length $n - 2$ constructed via the following iterative algorithm:

1. Initialize an empty sequence $b(T) = ()$ and set the current tree to $T_1 = T$.
2. For each step i from 1 to $n - 2$:
 - Find the leaf (vertex of degree 1) in T_i that has the **smallest label**. Let this vertex be v_i .
 - Let p_i be the unique neighbor of v_i in T_i .
 - Append p_i to the sequence, so that $b(T) = (p_1, \dots, p_i)$.
 - Remove the vertex v_i and its incident edge from T_i to form a new tree T_{i+1} .
3. The algorithm terminates after $n - 2$ steps when the remaining tree T_{n-1} consists of exactly two vertices. The final sequence is $b(T) = (p_1, p_2, \dots, p_{n-2}) \in \{1, 2, \dots, n\}^{n-2}$.

2. Proof of Cayley's Theorem

To prove that there are exactly n^{n-2} distinct labeled trees on the vertex set $\{1, 2, \dots, n\}$, we will establish that the Prüfer encoding function $b : \mathcal{T}_n \rightarrow \mathcal{P}_n$ is a ****bijection****, where:

- $\mathcal{T}_n$ is the set of all labeled trees on $\{1, 2, \dots, n\}$.
- $\mathcal{P}_n = \{1, 2, \dots, n\}^{n-2}$ is the set of all sequences of length $n-2$ using elements from $\{1, 2, \dots, n\}$.

Step 1: Well-Definedness and Cardinality of $\mathcal{P}_n$

By construction, every tree $T \in \mathcal{T}_n$ maps to a unique sequence $b(T)$ of length $n - 2$. The total number of such sequences is a basic combinatorial product. Since there are n independent choices for each of the $n - 2$ positions in the sequence, we have:

$$|\mathcal{P}_n| = \underbrace{n \times n \times \dots \times n}_{n-2 \text{ times}} = n^{n-2}$$

Step 2: Reconstructibility (Injectivity and Surjectivity)

To prove that b is a bijection, we show that any sequence $P = (p_1, p_2, \dots, p_{n-2}) \in \mathcal{P}_n$ can be uniquely decoded into a tree $T \in \mathcal{T}_n$.

The decoding algorithm relies on a fundamental structural property: *a vertex $k \in \{1, 2, \dots, n\}$ is a leaf in the current tree if and only if it does not appear in the remaining part of the Prüfer sequence.*

1. Maintain a set of available vertices $V = \{1, 2, \dots, n\}$ and an empty set of edges $E = \emptyset$.
2. For each element p_i in the sequence P (from $i = 1$ to $n - 2$):

- Identify the smallest vertex $v_i \in V$ that is ****not present**** in the remaining subsequence $(p_i, p_{i+1}, \dots, p_{n-2})$.
 - Add the edge (v_i, p_i) to E .
 - Remove v_i from V .
3. After $n - 2$ iterations, the sequence is exhausted, and exactly two vertices remain in the set V , say u and w . Add the final edge (u, w) to E .

Because this decoding algorithm uniquely determines a specific edge set E without any ambiguity for any given sequence P , the mapping b is invertible. Thus, b is a perfect bijection.

Conclusion

Since a bijection exists between the set of trees $\mathcal{T}_n$ and the set of sequences $\mathcal{P}_n$, their cardinalities must be equal:

$$|\mathcal{T}_n| = |\mathcal{P}_n| = n^{n-2}$$

This completes the proof of Cayley's Theorem. □