

CHAPTER 2

TREES

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1 Definitions- Examples

1.1 Definitions:

1. A graph G is *acyclic or forest* if it contains no cycle.
2. A graph G is a *tree* if G is connected and acyclic.

Remark:

A graph is acyclic if and only if each of its connected components is a tree.

1.2 Examples:

The trees on at most 5 vertices are, up to isomorphism, the following 8 graphs (and the null graph).

1. Tree with one vertex $T = (\{v\}, \emptyset)$



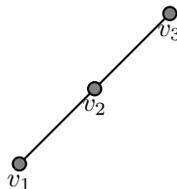
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2. Tree with 2 vertices $T = (\{u, v\}, \{\{u, v\}\})$



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3. Tree with 3 vertices $T = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\})$

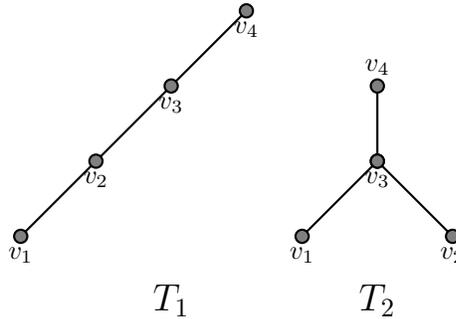


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4. Tree with 4 vertices

$$T_1 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}),$$

$$\text{and } T_2 = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\}).$$



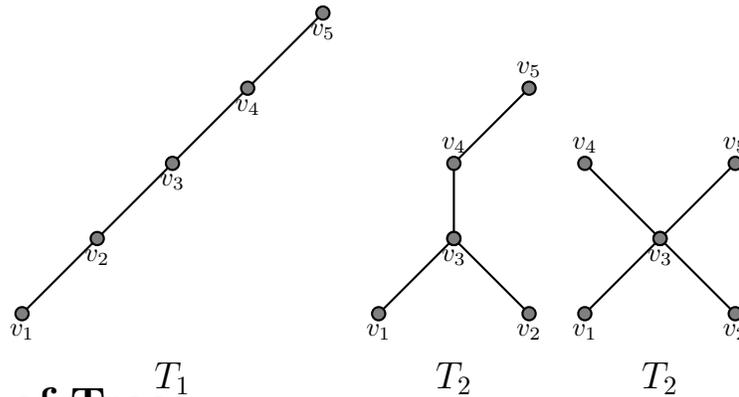
5. Tree with 5 vertices

$$T_1 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}),$$

$$T_2 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}),$$

and

$$T_3 = (\{v_1, v_2, v_3, v_4, v_5\}, \{\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_3\}, \{v_5, v_3\}\}).$$



2 Properties of Trees

2.1 Remarks:

Given a graph $G = (V, E)$ and a set S of edges of G , we can consider the subgraph of G whose edge set is S and whose vertex set is the set of all ends of edges of S . This subgraph is denoted $G[S]$, and called "the edge-induced subgraph $G[S]$ ".

Note that: an edge-induced subgraph is simply a subgraph without isolated vertices.

Lemma 2.1

Let $G = (V, E)$ be a graph and $u \neq v$ two vertices.

If G has two distinct paths P_1 and P_2 from u to v , then G contains a cycle.

Proof.

Consider the edge-induced subgraph $H = G[E(P_1) \cup E(P_2)]$. Each vertex of H is of degree ≥ 2 , then H contains a cycle (So G also).

Proposition 2.2

In a tree, any two distinct vertices are connected by exactly one path.

Proof.

Let $T = (V, E)$ be a tree and $u \neq v \in V$. As T is connected, then T has at least one path P from u to v . If T has a path $P' \neq P$, from u to v , then by Lemma 2.1, T contains a cycle; contradiction.

Remarks 2.3

1. *As any graph in which all degrees are at least two contains a cycle, then every tree has a vertex of degree at most one.*
2. *Thus, any tree T on $n \geq 2$ vertices has at least a vertex of degree 1. Each such a vertex is called "a leaf of T ".*
3. *Given a graph $G = (V, E)$ and a vertex x of G such that: $d_G(x) = 1$. Then:*
 - (a) *(G is connected) if and only if ($G - x$ is connected).*
 - (b) *(G is acyclic) if and only if ($G - x$ is acyclic).*
4. **Note** *Thus, given a tree $T = (V, E)$ on $n \geq 2$ vertices, then: for each leaf x of T , $T - x$ is a tree.*

Lemma 2.4

Given an acyclic graph $G = (V, E)$ with at least one edge (i.e nonempty acyclic graph), then G has at least two vertices of degree 1.

Proof.

Consider a path $P = (u_1, \dots, u_p)$ of G , with maximum length (in particular: $p \geq 2$). Then, $d(u_1) \geq 1$ and $d(u_p) \geq 1$.

If $d(u_1) > 1$ (resp. $d(u_p) > 1$), then there is $y \in N(u_1) \setminus \{u_2\}$ (resp. $y \in N(u_p) \setminus \{u_{p-1}\}$).

We distinguish two cases as follows:

Firstly: If $y = u_i$, for some $i \in \{1, \dots, p\}$, then $p \geq 3$ and $C = (u_1, u_2, \dots, u_i = y, u_1)$ is a cycle of G (resp. $C = (u_i = y, u_{i+1}, \dots, u_p, u_i = y)$ is a cycle of G); contradiction.

Secondly: If $y \notin \{u_j; 1 \leq j \leq p\}$, then $P' = (y, u_1, \dots, u_p)$ is a path of G and its length l verifies: $l > \text{length of } P$; contradiction.

Thus, $d_G(u_1) = d_G(u_p) = 1$.

Proposition 2.5

Every tree on $n \geq 2$ vertices, has at least two leaves.

Proof.

As a tree T on $n \geq 2$ vertices is an acyclic graph with at least one edges, we conclude by Lemma 2.4

Proposition 2.6

For every tree T on $n \geq 1$ vertices, we have: $e(T) = n - 1$ (i.e: $e(T) = v(V) - 1$).

Proof.

By induction on n . If $n = 1$, then $T \simeq K_1$ and then $e(T) = 0 = v(V) - 1$.

Let $n \geq 1$ be an integer and assume that for every tree T on $n \geq 1$ vertices, $e(T) = n - 1$. Consider a tree $T' = (V, E)$ with: $v(T') = |V| = n + 1$. By Proposition 2.5, we may consider a leaf u of T' . As $T' - u$ is a tree on n vertices, then by hypothesis (of induction); $e(T' - u) = v(T' - u) - 1$. But, $e(T' - u) = e(T') - 1$ (because $d_{T'}(u) = 1$) and $v(T' - u) = |V(T') \setminus \{u\}| = v(T') - 1 = n$. So, $e(T') - 1 = n - 1$. Thus, $e(T') = n = v(T') - 1$.

3 Characterization of Trees

Lemma 3.1

Let $G = (V, E)$ be a graph without isolated vertex such that: $|V| = n \geq 2$ and $|E| = n - 1$. Then G has at least two vertices of degree 1.

Proof.

Let $A = \{u \in V : d(u) = 1\}$ and $B = V \setminus A$. $2|E| = \sum_{u \in V} d(u) = \sum_{u \in A} d(u) + \sum_{u \in B} d(u)$.

So, $2(n - 1) = |A| + \sum_{u \in B} d(u)$. As: $\forall u \in B, d(u) \geq 2$, then: $2(n - 1) \geq |A| + 2|B| = |A| + 2(n - |A|)$.

Thus, $2n - 2 \geq -|A| + 2n$ and then: $|A| \geq 2$.

Theorem 3.2

Let $G = (V, E)$ be a graph with: $|V| = n \geq 1$. The following assertions are equivalent.

1. G is a tree.
2. G is connected and $|E| = n - 1$.
3. G is acyclic and $|E| = n - 1$.

Proof.

- "1. \Rightarrow 2." If G is a tree, then G is connected (by definition) and $|E| = n - 1$ by Proposition 2.6.
- "2. \Rightarrow 3." By induction on n (for " G is acyclic").
 - \rightarrow For $n = 1$ or $n = 2$, the result is immediate.
 - \rightarrow For $n \geq 2$ such that the result is true for n . Consider a graph $G' = (V', E')$ such that G' is connected, $|V'| = n + 1$ and $|E'| = n$. By Lemma 3.1, consider a vertex u of G' such that: $d_{G'}(u) = 1$ (indeed there are at least two such vertices). Clearly, $G' - u$ is connected with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = |E'| - 1 = n - 1$. Thus, by hypothesis (of induction), $(G' - u)$ is acyclic. As, $d_{G'}(u) = 1$, then G' is also acyclic.
- "3. \Rightarrow 1." By induction on n .
 - \rightarrow For $n = 1$, the result is trivial.
 - \rightarrow Assume that the result (i.e. this implication) is true for some $n \geq 1$ and consider a graph $G' = (V', E')$ such that G' is acyclic, $|V'| = n + 1$ and $|E'| = n$. By Lemma 2.4, consider a vertex u of G' such that: $d_{G'}(u) = 1$. Thus, $G' - u$ is (also) acyclic with: $v(G' - u) = n$ and $e(G' - u) = e(G') - 1 = n - 1$. Thus, by hypothesis (of induction), $(G' - u)$ is a tree, then is connected. Thus, G' is also connected (and then G' is a tree, because G' is acyclic by hypothesis).

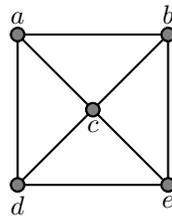
4 Spanning trees

Definition 4.1

1. Given a graph $G = (V, E)$, a subgraph of G which is a tree is called "a subtree of G ".
2. A spanning subgraph of G which is a tree is called spanning tree of G .

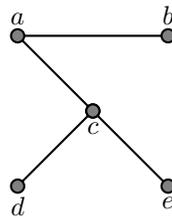
Example 4.2

Let $G = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, e\}, \{e, d\}, \{e, c\}, \{d, c\}\})$ be a graph.



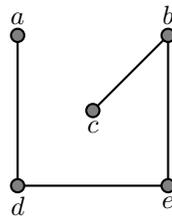
G

Then $T_1 = (\{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{e, c\}, \{d, c\}\})$ is spanning tree of G .



T_1

Then $T_2 = (\{a, b, c, d, e\}, \{\{a, d\}, \{e, d\}, \{b, e\}, \{b, c\}\})$ is spanning tree of G .



T_2

Remarks 4.3

1. Recall that a result in chapter 2: Given an edge e of a graph G , (e is a bridge of G) if and only if (e does not lie on any cycle of G).
2. A bridge of G which is also called "a cut edge of G " (See Book of "Bondy and Murty").

Theorem 4.4

A graph is connected if and only if it has a spanning tree.

Proof.

- " \Leftarrow " If a graph G has a spanning tree T , then two distinct vertices of G (and then of T) are connected by a path in T , which is also a path in G ; so G is connected.
- " \Rightarrow " Consider a connected graph $G = (V, E)$ which is not tree. Given an edge e of a cycle of G , $G - e = (V, E \setminus \{e\})$ is a spanning subgraph of G which is also connected (because by remark e is not a bridge of G). By repeating this process of deleting edges in cycles until every edge which remains does not lie in a cycle and then it is a cut edge (i.e bridge), we obtain a spanning subgraph T of G such that: T is connected and acyclic. So, T is a spanning tree of G .

Corollary 4.5

If $G = (V, E)$ is a connected graph with $|V| = n \geq 1$, then $|E| \geq n - 1$.

Proof.

By Theorem 4.4, G has a spanning tree T . By Theorem 3.2, $|E(T)| = n - 1$. As, $E(T) \subset E(G) = E$, then: $|E| \geq n - 1$.

5 Characterization of bipartite Graphs

Remarks 5.1

1. A graph G is a bipartite if and only if each of its connected components is bipartite.
2. A graph G contains an odd cycle if and only if one of its connected components contains an odd cycle.

Theorem 5.2

A graph is bipartite if and only if it contains no odd cycles.

Proof.

By remark 5.1, it suffices to prove the theorem in the case of connected graphs.

- " \Rightarrow " Let $G = G[X, Y]$ be a connected bipartite graph. The vertices of any path in G belong alternately to X and to Y . Thus, all paths connecting 2 vertices in different parts (one in X and the other in Y) is of odd length, and all paths connecting 2 vertices in the same part (X or Y) are of even length.

It follows that every cycle $(\alpha_1, \alpha_2, \dots, \alpha_p, \alpha_1)$ in G is of even length (because α_1 and α_p are in different parts, so $l(\alpha_1, \alpha_2, \dots, \alpha_p)$ is odd and then $l(\alpha_1, \alpha_2, \dots, \alpha_p, \alpha_1)$ is even).

- " \Leftarrow " Conversely; consider a connected graph G without odd cycles: By Theorem 4.4, G has a spanning tree $T = (V, E_1)$ (where $G = (V, E)$). We may assume that: $|V| \geq 2$.

Let $x \in V$. By Proposition 2.2, for each $v \in V \setminus \{x\}$, there is only one path P_v , in T , connecting v to x .

Let: $X = \{x\} \cup \{v \in V \setminus \{x\} : l(P_v) \text{ is even}\}$ and $Y = V \setminus X$.

Note That: $Y = \{v \in V \setminus \{x\} : l(P_v) \text{ is odd}\}$, and $Y \neq \emptyset$ (because $N_T(x) \neq \emptyset$ and $N_T(x) \subseteq Y$).

Thus, $\{X, Y\}$ is a partition of V (because $X \neq \emptyset$, $Y \neq \emptyset$, and $X \cap Y = \emptyset$).

Fact 1: $T[X]$ is an empty graph.

Indeed:

$\rightarrow \forall \alpha \in X$, $\{\alpha, x\} \notin E(T)$ because $N_T(x) \subseteq Y$.

\rightarrow Assume that $T[X]$ is a nonempty graph. So, there is an edge $\{\alpha, \beta\} \subseteq X \setminus \{x\}$.

Assume, for example, that $l(P_\alpha) \leq l(P_\beta)$ and let $P_\alpha = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha)$. As, $l(P_\alpha) \leq l(P_\beta)$, then $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So, $P'_\alpha = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha, \beta)$ is a path in T connecting x to β . It ensues, that: $l(P_\beta) = l(P'_\alpha) = l(P_\alpha) + 1$. Then $l(P_\beta)$ is odd (because $l(P_\alpha)$ is even; $(\alpha \in X)$); contradicts the fact that $\beta \in X$.

Fact 2: $T[Y]$ is an empty graph.

Indeed:

Assume that $T([Y]$ is a nonempty graph and consider an edge $\{\alpha, \beta\}$ of $T([Y]$ with:

$l(P_\alpha) \leq l(P_\beta)$. Denote: $P_\alpha = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha)$. Then, $\beta \notin \{\alpha_i; 1 \leq i \leq p\}$. So,

$P_\beta = (\alpha_1 = x, \alpha_2, \dots, \alpha_p = \alpha, \beta)$ is a path in T connecting x to β . So, $l(P_\beta) = l(P_\alpha) + 1$.

Contradiction.

\rightarrow Thus, T is bipartite graph with the partition $\{X, Y\}$.

Now, we will prove that G is bipartite with $\{X, Y\}$ as a bipartition.

For this, consider an edge $e \in E \setminus E_1 = E(G) \setminus E(T)$. Denote: $e = \{u, v\}$.

Let $P = (\alpha_1 = u, \alpha_2, \dots, \alpha_q = v)$ the unique path in T , from u to v . By hypothesis, the cycle $C = P + e$ is of even length. Thus, $l(P) = l(C) - 1$ is odd. As P is a path of the bipartite graph $T = T[X, Y]$, then u and v belong to distinct parts (i.e one is in X and the other is in Y).

Thus, every edge e' of G is such that: one end of e' is in X and the other end is in Y .

So G is bipartite and $G = G[X, Y]$.

Corollary 5.3

Every tree is a bipartite graph.

Proof.

As a tree is without cycles, this corollary is an immediate consequence of Theorem 5.2.

6 Exercises of Trees

N.B: All the graphs considered here are nontrivial

Exercise 6.1

Let F be a forest of order p and size q having k connected components. Obtain an expression for q in terms of p and k .

Exercise 6.2

1. (a) Find all the trees, up to isomorphism, T is a regular graph.
 (b) Find all the trees, up to isomorphism, T is a complete bipartite graph $K_{p,q}$ (where $p, q \geq 1$).
2. (a) Find all the trees T , up to isomorphism, on $\{1, 2, \dots, n\}$ where $n \geq 3$, and T has exactly $n - 1$ leaves.
 (b) Find all the trees T , up to isomorphism, on $\{1, 2, \dots, n\}$ where $n \geq 3$, and T has exactly 2 leaves.

Exercise 6.3

Given an integer $n \geq 2$, show that an increasing sequence $D = (d_1, d_2, \dots, d_n)$ of positive integers is the degree sequence of some tree if and only if $\sum_{i=1}^n d_i = 2n - 2$.

(Hint: Prove that if $n \geq 3$, and $\sum_{i=1}^n d_i = 2n - 2$, then $d_1 = 1$ and $d_n > 1$, and use an induction on n .)

Exercise 6.4

Show that any tree T has at least $\Delta(T)$ leaves.

Exercise 6.5

Let T be a tree with $v(T) = n \geq 3$, and denote $x_i = |\{v \in V(T) : d_T(v) = i\}|$.

1. Show that $\sum_{i=3}^{n-1} (i - 2)x_i = x_1 - 2$,
2. Count the number of nonisomorphic trees that have 5 leaves and no vertices of degree 2.

Exercise 6.6

Let G be a connected graph.

1. Let T be a spanning tree of G . Show that every cycle of G has an edge that is in the complement \bar{T} .
2. Let $e \in E(G)$. Show that e is a bridge of G if and only if e belongs to every spanning tree of G .

Exercise 6.7

Let T be a tree of order k . Show that any graph G with $\delta(G) \geq k - 1$ has a subgraph isomorphic to T .

(Hint: Use an induction on k).

Exercise 6.8

Let T be a tree of order $n \geq 2$.

1. Let $u \in V(T)$, and suppose that $P : (v_1 = u, v_2, \dots, v_p = v)$ is a maximal path in T such that u is an end of P . Show that v is a leaf of T .
2. Show that there is a vertex v of T such that $N_T(v)$ consists of leaves excepting possibly one neighbour.

Exercise 6.9

Let $A \subseteq \mathbb{N}$ such that $0 \notin A$ and $1 \in A$. Show that there is a tree T such that $A = \{d_T(v) : v \in V(T)\}$.

(Hint: Use an induction on $n = |A|$).

Exercise 6.10

1. Define the Prüfer code $b(T) = (p_1, p_2, \dots, p_{n-2})$ of a tree T of order n .
2. Prove Cayley's Theorem: There are exactly n^{n-2} trees on $\{1, 2, \dots, n\}$.