# Chapter Two
# The Inclusion-Exclusion Principle

The sum principle for counting is the simplest of the basic counting principles. It states that if $A_1, A_2, \ldots, A_n$ are finite sets that are pairwise disjoint, then :

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

The Inclusion-Exclusion Principle, in its simplest form, provides a formula to calculate $|A_1 \cup A_2 \cup \cdots \cup A_n|$ when we allow the sets $A_1, A_2, \ldots, A_n$ to overlap.

In what follows, we assume that $U$ is a given finite universal set and that $A_1, A_2, \ldots, A_n$ are subsets of $U$. For each $i = 1, 2, \ldots, n$, we define :

$$\alpha_i = \sum_{\{j_1, j_2, \ldots, j_i\} \subseteq \{1,2,\ldots,n\}} |A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}|$$

where the sum is taken over all possible subsets of indices $\{j_1, j_2, \ldots, j_i\} \subseteq \{1, 2, \ldots, n\}$.

### Theorem (2.1) (The Inclusion-Exclusion Principle)

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \alpha_1 - \alpha_2 + \cdots + (-1)^{n-1}\alpha_n \tag{1}$$

## Proof

Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. When calculating $|A_1 \cup A_2 \cup \cdots \cup A_n|$, $x$ is counted exactly once. The situation is different when calculating each $\alpha_1, \alpha_2, \ldots, \alpha_n$. We will prove that the contribution of $x$ to the calculation of the number $\alpha_1 - \alpha_2 + \cdots + (-1)^{n-1}\alpha_n$ is equal to 1.

Assume that $x$ belongs only to $m$ sets among $A_1, \ldots, A_n$. Therefore, the contribution of $x$ to the calculation of $\alpha_1 = |A_1| + |A_2| + \cdots + |A_n|$ is $m$. Also, the contribution of $x$ to the calculation of $\alpha_2$ is $\binom{m}{2}$, because the contribution of $x$ to $|A_i \cap A_j|$ is 0 if $\{i, j\} \not\subseteq \{j_1, \ldots, j_m\}$ and 1 if $\{i, j\} \subseteq \{j_1, \ldots, j_m\}$.

Similarly, the contribution of $x$ to the calculation of $\alpha_i$ is $\binom{m}{i}$ for each $1 \leq i \leq n$. Thus, the contribution of $x$ to $\alpha_1 - \alpha_2 + \cdots + (-1)^{n-1}\alpha_n$ is :

$$\binom{m}{1} - \binom{m}{2} + \cdots + (-1)^{m-1}\binom{m}{m}$$

since $\binom{m}{i} = 0$ for each $m < i \leq n$. However, from the Binomial Theorem, we know that :

$$(1-1)^m = \binom{m}{0} - \binom{m}{1} + \cdots + (-1)^m \binom{m}{m} = 0$$

Therefore :

$$\binom{m}{1} - \binom{m}{2} + \cdots + (-1)^{m-1} \binom{m}{m} = \binom{m}{0} = 1$$

This completes the proof. ■

In many problems, we calculate the number of elements that do not belong to any of the sets $A_1, A_2, \ldots, A_n$ using the following result of the Inclusion-Exclusion Principle.

## Corollary (2.1)

If $U$ is a finite universal set and $A_1, A_2, \ldots, A_n$ are subsets of $U$, then :

$$|U - (A_1 \cup A_2 \cup \cdots \cup A_n)| = |U| - \alpha_1 + \alpha_2 - \cdots + (-1)^n \alpha_n \tag{2}$$

Now, we rely on the Inclusion-Exclusion Principle and its corollary to present a collection of various theorems and examples.

## Theorem (2.2)

The number of surjective (onto) functions from the set $A = \{a_1, a_2, \ldots, a_m\}$ to the set $B = \{b_1, b_2, \ldots, b_n\}$, where $m \geq n$, is equal to :

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m = \sum_{k=0}^{n-1}(-1)^k \binom{n}{k}(n-k)^m$$

**Proof** Let $U$ be the set of functions from $A$ to $B$. For each $1 \leq k \leq n$, let $A_k = \{f \in U : b_k \notin R(f)\}$, where $R(f)$ denotes the range of $f$. Thus, we need to calculate the number $|U - (A_1 \cup A_2 \cup \cdots \cup A_n)|$.

It is clear that $|U| = n^m$. Now, we calculate $\alpha_1$ from the relation $\alpha_1 = |A_1| + |A_2| + \cdots + |A_n|$. From the definition of $A_k$, it follows that $|A_k|$ is the number of functions from $A$ to $B - \{b_k\}$. Thus, $|A_k| = (n-1)^m$ for each $1 \leq k \leq n$. Therefore, $\alpha_1 = \binom{n}{1}(n-1)^m$.
To calculate $\alpha_2$ :
We notice that $|A_i \cap A_j|$, where $1 \leq i < j \leq n$, is equal to the number of functions from $A$ to $B - \{b_i, b_j\}$. Thus, $|A_i \cap A_j| = (n-2)^m$, and consequently :

$$\alpha_2 = \binom{n}{2}(n-2)^m$$

Similarly, we find that $\alpha_k = \binom{n}{k}(n-k)^m$ for each $1 \leq k \leq n$. Therefore :

$$\begin{aligned}|U - (A_1 \cup A_2 \cup \cdots \cup A_n)| &= |U| - \alpha_1 + \alpha_2 - \cdots + (-1)^n \alpha_n \\ &= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m \\ &= \sum_{k=0}^{n-1}(-1)^k \binom{n}{k}(n-k)^m\end{aligned}$$

## Example (2.3)

Find the number of integers $x$ such that $1 \le x \le 500$, where $x$ is not divisible by 5, not divisible by 6, and not divisible by 8.

**Solution** Let $U = \{1, 2, \ldots, 500\}$, and let $A_1 = \{x \in U : 5|x\}$, $A_2 = \{x \in U : 6|x\}$, and $A_3 = \{x \in U : 8|x\}$. We want to calculate the number $|U - (A_1 \cup A_2 \cup A_3)|$. We note that :

$$|A_1| = \left\lfloor \frac{500}{5} \right\rfloor = 100, \quad |A_2| = \left\lfloor \frac{500}{6} \right\rfloor = 83, \quad |A_3| = \left\lfloor \frac{500}{8} \right\rfloor = 62$$

As is known, $a|n$ and $b|n$ if and only if $\operatorname{lcm}(a, b)|n$. Therefore, we find :

$$\begin{aligned}
|A_1 \cap A_2| &= \left\lfloor \frac{500}{\operatorname{lcm}(5,6)} \right\rfloor = \left\lfloor \frac{500}{30} \right\rfloor = 16 \\
|A_1 \cap A_3| &= \left\lfloor \frac{500}{\operatorname{lcm}(5,8)} \right\rfloor = \left\lfloor \frac{500}{40} \right\rfloor = 12 \\
|A_2 \cap A_3| &= \left\lfloor \frac{500}{\operatorname{lcm}(6,8)} \right\rfloor = \left\lfloor \frac{500}{24} \right\rfloor = 20 \\
|A_1 \cap A_2 \cap A_3| &= \left\lfloor \frac{500}{\operatorname{lcm}(5,6,8)} \right\rfloor = \left\lfloor \frac{500}{120} \right\rfloor = 4
\end{aligned}$$

Therefore :

$$\begin{aligned}
|U - (A_1 \cup A_2 \cup A_3)| &= |U| - (\alpha_1 - \alpha_2 + \alpha_3) \\
&= 500 - (100 + 83 + 62) + (16 + 12 + 20) - 4 = 299
\end{aligned}$$

## Example (2.4)

Calculate $\phi(40)$, which is the value of Euler's totient function for the number 40.

**Solution** Let $U = \{1, 2, \ldots, 40\}$. We want to find the number of integers in $U$ that are relatively prime to 40. Since $40 = 2^3 \times 5$, an integer $x$ is relatively prime to 40 if and only if it is not divisible by 2 and not divisible by 5.
Let $A_1 = \{x \in U : 2|x\}$ and $A_2 = \{x \in U : 5|x\}$. Then :

$$\phi(40) = |U| - (|A_1| + |A_2|) + |A_1 \cap A_2| = 40 - \left(\frac{40}{2} + \frac{40}{5}\right) + \frac{40}{10} = 40 - 28 + 4 = 16$$

In general, if the prime factorization of $n$ is $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, then an integer $x$ is relatively prime to $n$ if and only if $x$ is not divisible by any of the prime factors $p_1, p_2, \ldots, p_k$.
For each $1 \le i \le k$, let $A_i = \{x \in \{1, 2, \ldots, n\} : p_i|x\}$. Then $\phi(n) = |U - (A_1 \cup A_2 \cup \cdots \cup A_k)|$. We notice that $|A_i| = \frac{n}{p_i}$, $|A_i \cap A_j| = \frac{n}{p_i p_j}$, and so on. Therefore :

$$\begin{aligned}
\phi(n) &= n - \sum \frac{n}{p_i} + \sum \frac{n}{p_i p_j} - \cdots + (-1)^k \frac{n}{p_1 p_2 \ldots p_k} \\
&= n \left(1 - \sum \frac{1}{p_i} + \sum \frac{1}{p_i p_j} - \cdots + (-1)^k \frac{1}{p_1 p_2 \ldots p_k}\right) \\
&= n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right)
\end{aligned}$$

### Example (2.5)

Find the number of integer solutions to the equation $X_1 + X_2 + X_3 = 13$ subject to the conditions $0 \leq X_1 \leq 6$, $0 \leq X_2 \leq 9$, and $0 \leq X_3 \leq 3$.

**Solution** Let $U$ be the set of all non-negative integer solutions to the equation $X_1 + X_2 + X_3 = 13$.
Let $A_1$ be the set of integer solutions with $X_1 \geq 7$, $A_2$ with $X_2 \geq 10$, and $A_3$ with $X_3 \geq 4$. Thus, we need to calculate the number $|U - (A_1 \cup A_2 \cup A_3)|$.

It is clear that $|U| = \binom{3-1+13}{13} = \binom{15}{13} = 105$. Similarly, we find that :

$$\begin{aligned} |A_1| &= \binom{3-1+13-7}{13-7} = \binom{8}{6} = 28 \\ |A_2| &= \binom{3-1+13-10}{13-10} = \binom{5}{3} = 10 \\ |A_3| &= \binom{3-1+13-4}{13-4} = \binom{11}{9} = 55 \end{aligned}$$

We also note that $|A_1 \cap A_2| = 0$ because $7 + 10 = 17 > 13$.
Furthermore :

$$\begin{aligned} |A_1 \cap A_3| &= \binom{3-1+13-7-4}{13-7-4} = \binom{4}{2} = 6 \\ |A_2 \cap A_3| &= 0 \quad (\text{since } 10 + 4 = 14 > 13) \\ |A_1 \cap A_2 \cap A_3| &= 0 \end{aligned}$$

Therefore, the number of required solutions is :

$$105 - (28 + 10 + 55) + (0 + 6 + 0) - 0 = 18$$

From the definition of the union of sets, it follows that the Inclusion-Exclusion Principle provides the number of elements that belong to at least one of the sets $A_1, A_2, \ldots, A_n$.

## Exercises

1. A survey of 100 students showed that 32 students are enrolled in Course A, 44 students are enrolled in Course B, 47 students are enrolled in Course C, 11 students are enrolled in both Courses B and C, 12 students are enrolled in both Courses A and C, 12 students are enrolled in both Courses A and B, and 3 students are enrolled in all three courses. Find the number of students who are not enrolled in any of the three courses.
2. Tests were conducted on 200 groundwater samples to detect the presence of salts A, B, and C. It was found that 14 samples contain salt A, 10 samples contain salt B, 8 samples contain salt C, 6 samples contain salts A and B, 6 samples contain salts B and C, 4 samples contain salts A and C, and 2 samples contain salts A and B but do not contain salt C. Find the number of samples containing at least one of the three salts.

3. Find the number of integer solutions to the equation $X_1+X_2+X_3+X_4+X_5=20$ subject to :
   (a) $0 \leq X_i \leq 10$ for all $i=1,2,\ldots,5$.
   (b) $0 \leq X_i \leq 8$ for all $i=1,2,\ldots,5$.
4. Find the number of integer solutions to the equation $X_1+X_2+X_3+X_4=30$ subject to :
   (a) $0 \leq X_i \leq 8$ for all $i=1,2,3,4$.
   (b) $10 \leq X_i \leq 20$ for all $i=1,2,3,4$.
   (c) $0 \leq X_1 \leq 6,\ 0 \leq X_2 \leq 9,\ 0 \leq X_3 \leq 15,\ 0 \leq X_4 \leq 18$.
5. Find the number of integer solutions to the equation $X_1+X_2+X_3=30$ subject to :
$$5 \leq X_1 < 11, \quad 6 < X_2 \leq 14, \quad 10 \leq X_3 \leq 24$$
6. If $A=\{1,2,\ldots,999999\}$, what is the number of integers belonging to $A$ such that the sum of the digits of each number equals 15 ?
7. Find the number of multisets of size 15 chosen from the set $A=\{a_1,a_2,a_3\}$ such that the repetition of $a_1$ is less than 5, the repetition of $a_2$ is less than 7, and the repetition of $a_3$ is less than 6.
8. Find the number of integers $n$, where $1 \leq n \leq 2000$, such that :
   (a) $2 \nmid n,\ 3 \nmid n,\ 5 \nmid n$.
   (b) $2 \nmid n,\ 3 \nmid n,\ 5 \nmid n,\ 7 \nmid n$.
   (c) $2 \nmid n,\ 3 \nmid n,\ 5 \nmid n,\ 7 \nmid n$.
9. If $n=p_1^{a_1}p_2^{a_2}$ where both $p_1$ and $p_2$ are prime numbers, prove that :
$$\phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)$$
   Then calculate $\phi(135)$.