COMBINATORICS AND GRAPH THEORY-1-

January 26, 2026

CHAPTER 1

BASIC CONCEPTS IN GRAPH THEORY

1 Definitions and Examples

1.1 Definitions

1. **Graph :** A *graph* is an ordered pair $G = (V(G), E(G))$ (or simply $G = (V, E)$) where, $V(G)$ is a finite set, and $E(G)$ is a subset of $[V]^2$ ($[V]^2$ is the set of the pairs $\{u, v\}$ such that $u \neq v$).

2. Let $G = (V, E)$ be a graph
   (a) Each element of V is a *vertex* of G.
   (b) V is the *vertex set* of G.
   (c) Each element of E is an *edge* of G.
   (d) E is the *edge set* of G.

3. Occasionally, it is desirable to denote $V(G)$ the vertex set of a graph G and $E(G)$ its edge set. This is useful when we have two or more graphs under consideration.

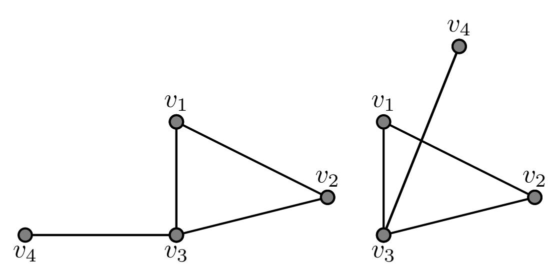
4. Let $G = (V, E)$ be a graph
   (a) The *order* of G denoted by: $|G|$ is the number $|V|$.
   (b) The *size* of G denoted by: $\|G\|$ is the number $|E|$.

5. Let $G = (V, E)$ be a graph. An edge $\{u, v\}$ is denoted simply uv.

6. It is convenient to represent a graph by a diagram.
   In such representation, we indicate the vertices by points (or small circles), and we represent the edges by line segments (or curves) joining the two appropriate points.

1.2 Examples

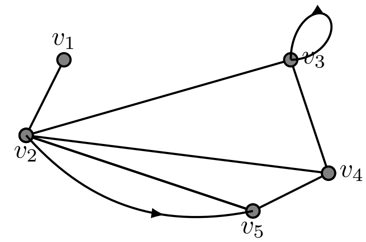
1. Let $G = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_1v_3, v_2v_3, v_3v_4\})$ be a graph



Two representations of the same graph G

Order of G is 4
Size of G is 4

2. Let $H = (\{v_1, v_2, v_3, v_4, v_5\}, \{v_1v_2, v_2v_3, v_3v_3, v_3v_4, v_2v_4, v_4v_5, v_2v_5, v_2v_5\})$ be a graph



Representation of H, the graph H is not a simple graph

Order of H is 5
Size of H is 8
We remak that, in this case, the graph H is not simple, because H has a double (multiple) edges (or because H has a loop).

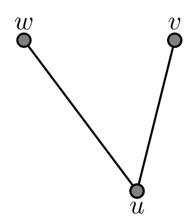
1.3 Definitions

1. Let $G = (V, E)$ be a graph.

   (a) For $x \neq y \in V$, we say that the vertices x and y are *adjacent* when $\{x, y\}$ is an edge. If not, the vertices x and y are *nonadjacent.*

   (b) If $e = \{x, y\}$ is an edge, x and y are the *ends* of e and x (and y) is *incident* with (to) the edge e.

   (c) If uv and uw are different edges (i.e: $v \neq w$) we say that the edges uv and uw are *adjacent.*
   Let $G = (\{u, v, w\}, \{uv, uw\})$ be a graph



Representation of G with the edges uv and uw are adjacent.

2. Let $G = (V, E)$ be a graph, and let v be a vertex of G.

   (a) Two adjacent vertices are *neighbours.*

   (b) The set of neighbours of vertex v, called the *neighborhood* of v; is denoted by: $N_G(v)$ (or simply $N(v)$).
   Let S be a subset of V. The *neighborhood of S*, denoted by $N(S)$, is the set of vertices in V that have an adjacent vertex in S. The elements of $N(S)$ are called the *neighbours* of S, noted that: $N(\{v\}) = N(v)$.

   (c) The *degree* of the vertex v is the number $\mid N_G(v) \mid$ denoted by: $d_G(v)$ or $deg(v)$ (or simply $d(v)$).
   A vertex v of the graph G is called *vertex even* or *vertex odd* according to the parity of

its degree.
A vertex v of the graph G is called *isolated vertex*, if $d_G(v) = 0$, and a vertex of degree 1 in G is called a *leaf.*

(d) The *maximum degree of the vertices* of G is denoted: $\Delta(G)$.

(e) The *minimum degree of the vertices* of G is denoted: $\delta(G)$.

(f) The *average degree of the vertices* of G denoted by: $d(G)$ such that, $d(G) = \frac{1}{n}\sum_{v \in V} d(v)$, where $n = |V| \geq 1$.

N. B: It is easily to see that: $\delta(G) \leq d(G) \leq \Delta(G)$.

3. Given a graph G, with the vertex set $V = \{v_1, v_2, ..., v_n\}$, the sequence $(d(v_1), ..., d(v_n))$ is called the *degree sequence* of G.

1.4 Remarks

1. Given a graph $G = (V, E)$, we denote $v(G) = |V|$ and $e(G) = |E|$.
2. The term "graph" always means 'finite graph', we call a graph with just one vertex *trivial* and all other graphs *nontrivial.*
3. Much of graph theory is concerned with the study of simple graphs.
4. The graph with no vertices (and then no edges) is the *null graph.*
   Unless otherwise specified, we consider *non null graphs* (i.e: $V(G) \neq \emptyset$).
5. Given a graph of order n, we can enumerate his vertices by: $v_1, v_2, ..., v_n$ such that, $d(v_1) \leq ... \leq d(v_n)$.
6. The increasing (or decreasing) sequence $(d(v_1), ..., d(v_n))$ is the *degree sequence* of G.

2 Vertex degrees

2.1 Properties of vertex degrees

Proposition 2.1 *Let G be a graph, $\delta(G) \leq d(G) \leq \Delta(G)$.*

Theorem 2.2 ***(Handshake lemma)***
For any graph G, the sum of the degrees of the vertices of G equals twice the number of edges of G. (i.e: $\sum_{v \in V} d(v) = 2|E|$, where $G = (V, E)$).

Proof.
Let $G = (V, E)$ be a graph and consider the sum $S = \sum_{v \in V} d(v)$. For $a \neq b \in V$, we count the edge $\{a, b\}$ **twice** if $\{a, b\} \in E$ (one in $d(a)$ and one in $d(b)$), and we don't count the edge $\{a, b\}$ if $\{a, b\} \notin E$. So, $S = 2|E|$.

Corollary 2.3

Every graph contains an even number of odd vertices.

Proof.

Let $G = (V, E)$ be a graph and consider $V(G) = A \cup B$ where, A (resp. B) is the set of even (resp. odd) vertices of G.

We have $\sum_{v \in V} d(v) = \sum_{v \in A} d(v) + \sum_{v \in B} d(v) = 2|E|$, hence $\sum_{v \in B} d(v) = 2|E| - \sum_{v \in A} d(v)$, then $\sum_{v \in B} d(v)$ is even. It ensues that $|B|$ is even. (**Note:** $\sum_{v \in \emptyset} d(v) = 0$).

Theorem 2.4 *(***Pigeonhole Principle***)*

Let S be a finite set with $|S| = n$, and let $S_1, ..., S_k$ be a partition of S into k subsets such that: $1 \leq k < n$. Then at least one subset S_i contains at least $(\lfloor \frac{n-1}{k} \rfloor + 1)$ elements (let y be a real number, $\lfloor y \rfloor$ is the greatest integer p, $p \leq y$, and $\lfloor y \rfloor$ is called the floor of y).

Proof.

By contradiction. If not: $\forall i \in \{1, ..., k\}$, $|S_i| \leq \lfloor \frac{n-1}{k} \rfloor$.

So, $|V| = \sum_{1 \leq i \leq k} |S_i| \leq k.\dfrac{n-1}{k} = n - 1 < n$.

Thus $|V| = n < n$; contradiction.

Corollary 2.5

Given a graph $G = (V, E)$ on $n \geq 2$ vertices, there are $x \neq y \in V$ such that: $d(x) = d(y)$.

Proof.

Given $G = (V, E)$ a graph, the first remark, if there is an isolated vertex x (i.e: $d(x) = 0$), then: $(\forall y \in V,\ d(y) \leq n - 2)$ and the second remark, if there is a vertex x such that $d(x) = n - 1$, then: $(\forall y \in V,\ d(y) \geq 1)$.

By the first remark and the second remark we deduce $(\forall v \in V, d(v) \in \{0, ..., n - 2\})$ or $(\forall v \in V,\ d(v) \in \{1, ..., n - 1\})$.

Thus, the n values: $d(v_1), ..., d(v_n)$ (where $V = \{v_1, ..., v_n\}$) are all in set A with: $|A| = n - 1$. So, we conclude by the **Pigeonhole Principle**.

Corollary 2.6 *(***Particular case of Pigeonhole Principle***)*

If we put n pigeons in k cages such that $k < n$, then at least one cage contains at least two pigeons.

2.2 Exercises

1. (a) Show that there is no graph with degree sequence: $(2, 3, 3, 4, 4, 5)$.
   (b) Show that there is no graph with degree sequence: $(2, 3, 4, 4, 4, 6, 6, 6, 9)$.
   (c) Show that there is no graph with degree sequence: $(1, 3, 3, 3)$.
   (d) Show that there is no graph with degree sequence: $(1, 2, 4, 5, 6, 6, 7, 8, 9)$.
   (e) Show that there is no graph with degree sequence: $(1, 2, 3, 4, 4)$.
   (f) Show that there is no graph with degree sequence: $(2, 3, 4, 5, 5, 5)$.

2. Show that, given a group of $n \geq 2$ students, there are at least two students (from this group) having the same number of friends (in the group).
3. We have 15 computers. Is it possible to connect each of them to exactly 3 others?
4. Let p, n two odd integers, such that $p < n$. We have n computers. Is it possible to connect each of them to exactly p others?

3 Subgraph

3.1 Definitions

1. **Subgraph:** A *subgraph* of a graph $G = (V(G), E(G))$ is a graph $H = (V(H), E(H))$ verifying:
   - $V(H) \subseteq V(G)$
   - $E(H) \subseteq E(G)$
2. If H is a subgraph of G, we say that G **contains** H (or that H is contained in G, and we write: $G \supseteq H$ (or $H \subseteq G$)).
3. Let $G = (V, E)$ be a graph.
   - A *spanning subgraph* of a graph G is a subgraph H of G such that: $V(H) = V$.
   - For $X \subseteq V$, the subgraph $(X, E \cap [X]^2)$ of G is called the subgraph of G *induced* by X; it's denoted: $G[X]$.
4. Let $G = (V, E)$ be a graph.
   - If $e \in E$, the subgraph $(V, E \setminus \{e\})$ of a graph G is denoted: $G - e$. (Thus $G - e$ is obtained, from G, by deleting the edge e).
   - If $v \in V$, the subgraph $G[V \setminus \{v\}]$ induced by $V \setminus \{v\}$ is denoted by: $G - v$. (Thus, $G - v$ is obtained by deleting from G the vertex v together with all the edges incident with v).
5. - A copy of a graph H in a graph G, is a subgraph of G which is isomorphic to H. Such a subgraph is then a H- subgraph of G.
   - For example a K_3-subgraph of G is a triangle of G.
6. An *embedding* of a graph H in a graph G is an isomorphism between H and a subgraph of G ($\exists X \subseteq V, G[X] \simeq H$).
7. - A *supergraph* of a graph G is a graph G' which contains G as a subgraph, that is: $(G' \supseteq G)$.
   - Note that each graph G is both a subgraph and supergraph of itself.
   - All other subgraphs H and supergraphs G' are *proper*; we write: $H \subset G$ or $G' \supset G$, respectively.

3.2 Remarks

1. Let $G = (V, E)$ be a graph, $e \in E$, and $v \in V$.
   - $G - e$ is called an edge-deleted subgraph of G.
   - $G - v$ is called a vertex-deleted subgraph of G.
   - **Note** That any subgraph H of G can be obtained by repeated applications of the basic operations of edge-deletion and vertex-deletion. (for instance, by first deleting the edges of G not in H and then deleting the vertices of G not in H).
2. Given a graph $G(V, E)$, if $e = \{u, v\} \in [V]^2 \setminus E$, the supergraph $(V, E \cup \{e\})$ of G is denoted by: $G + e$.
3. The following theorem due to Erdös (1964/1965), confirms that every graph has an induced subgraph whose minimum degree is relatively large.

Theorem 3.1 *(Erdös)*
Let G be a graph with $d(G) \geq 2k$; where $k \geq 1$ is an integer. Then, G has an induced subgraph H with: $\delta(H) \geq k + 1$.

Proof.
Let G be a graph with $d(G) \geq 2k$; where $k \geq 1$ is an integer, and consider an induced subgraph $H = G[X]$, where $X \subseteq V$ such that:

1. H is with the **largest** possible average degree, that is $d(H)$ is maximum among $d(K)$ where K is induced subgraph of G, $d(K) \leq d(H)$, in particular G is an induced subgraph of G.
2. $|X| = |V(H)|$ is **minimum** among $|V(L)|$ where L is induced subgraph of G with: $d(L) = d(H)$, we notice that: if $|V(K)| < |V(H)|$, then $d(K) < d(H)$.

- **Note** For each graph $K = (V(K), E(K))$, we denoted: $v(K) = |V(K)|$ and $e(K) = |E(K)|$.
- We will show that: $\delta(H) \geq k + 1$.

Fact 1: $v(H) > 1$.
Indeed: If not $v(H) = 1$, then $0 = \delta(H) = d(H)$. But, $d(H) \geq d(G)$, because G is an induced subgraph of G (So by 1. we have $d(H) \geq d(G)$), then $0 \geq d(G) \geq 2k \geq 2$; contradiction.
Fact 2: $\forall x \in V(H), d_H(x) \geq k + 1$.
Indeed: Suppose by contradiction that $\exists x \in V(H) : d_H(x) \leq k$.
Consider the subgraph: $H' = H - x = H[X \setminus \{x\}]$

- **Clearly**, $H' = G[X \setminus \{x\}]$, and then H' is an induced subgraph of G.
- $d(H') = \dfrac{1}{v(H')} . \sum_{v \in V(H')} d_{H'}(v)$. Then,
  $d(H') = \dfrac{2e(H')}{v(H')} = \dfrac{2e(H')}{v(H) - 1}$.
- But, $d_H(x) \leq k$, then: $e(H') \geq e(H) - k$, so, $d(H') = \dfrac{2e(H')}{v(H) - 1} \geq \dfrac{2(e(H) - k)}{v(H) - 1}$.
  But, $2k \leq d(G)$, then $d(H') \geq \dfrac{2e(H) - d(G)}{v(H) - 1} \geq \dfrac{2e(H) - d(H)}{v(H) - 1} = d(H)$.
  Thus, $d(H') \geq d(H)$; contradiction ($v(H') < v(H)$).

4 Walks-Paths-Cycles

.

Definition 4.1

Consider a graph $G = (V, E)$.

1. - *A path P of G from u to v (where $u, v \in V$) is a sequence of vertices $u_0 = u, ..., u_k = v$ such that: $\forall i < k, \{u_i, u_{i+1}\} \in E(G)$, and all the u_i are distinct vertices.*
   - *The length is $l(P)$ the number of edges it uses. (Here, $l(P) = k$).*
   - *P is a uv-path of length k.*
2. **Incidence and Adjacency matrices**
   - *Let $G = (V, E)$ be a graph where: $V = \{v_1, ..., v_n\}$. The adjacency matrix of G is the (n, n) matrix $A_G = (a_{ij})_{1 \leq i,j \leq n}$, where: $a_{ij} = 1$, if $\{v_i, v_j\} \in E$, and $a_{ij} = 0$, if not.*
   - *Let $G = (V, E)$ be a graph where: $V = \{v_1, ..., v_n\}$ and $E = \{e_1, ..., e_m\}$. The incidence matrix of G is the (n, m) matrix $M_G = (m_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$, where: $m_{ij} = 1$, if $v_i \in e_j$, and $m_{ij} = 0$, if not.*
3. *A cycle C of G is a sequence of vertices $u_0, ..., u_k$ forming a u_0u_k-path such that: $\{u_0, u_k\} \in E(G)$ (where $k \geq 2$). We also denote $(u_0, ..., u_k, u_0)$ this cycle.*
4. *Consider a graph $G = (V, E)$.*
   - *Paths in G do not contain repeated vertices or edges.*
   - *Let $u, v \in V$ be a vertices, walk from u to v in G is any sequences of vertices $u = u_0, ..., u_k = v$ such that: $\forall i < k, \{u_i, u_{i+1}\} \in E(G)$.*
   - *A walk in G is any sequences of vertices $u_0, ..., u_k$ such that: $\forall i < k, \{u_i, u_{i+1}\} \in E(G)$. Thus in a walk, edges and vertices may be repeated.*
   - *The length of this walk is the number of its edges (here: k).*
   - *The trail is a walk w where all its edges are distinct.*

Proposition 4.2

*Let $u \neq v$ be a two vertices of a graph $G = (V, E)$.
If there is a walk $(u_0 = u, ..., u_k = v)$ from u to v, then we can extract a path from u to v: $u_{i_1} = u, ..., u_{i_p} = v$.*

Proof.

- Consider a walk $P = (\alpha_1 = u, ..., \alpha_q = v)$ which is extract from the initial walk $(u_0 = u, ..., u_k = v)$ and which is with **minimum** length (among all extract walks $(\beta_1 = u, ..., \beta_q = v)$). **Note that** the initial walk is extract from itself.
- **Fact** P is a path.
  Indeed: Assume by contradiction that there are $1 \leq i < j \leq p$ such that ; $\alpha_i = \alpha_j$. Thus, $P' = (\alpha_1 = u, ..., \alpha_{i-1}, \alpha_i = \alpha_j, \alpha_{j+1}, ..., \alpha_q = v)$ is an extract walk with: $l(P') < l(P)$. Contradiction.

Proposition 4.3

Let $A_G = (a_{ij})$ be the adjacency matrix of a graph $G = (V, E)$ where $V(G) = \{v_1, ..., v_n\}$. For any integer $k \geq 1$, let $A_G^k = (a_{ij}^{[k]})$. Then for each integer $k \geq 1$, we have: $\forall i, j \in \{1, ..., n\}$; $a_{ij}^{[k]}$ is the number of walks of length k from v_i to v_j.

Proof.

By induction on k.

- For $k = 1$, [there is a walk of length 1 from v_i to v_j] if and only if [$\{v_i, v_j\} \in E(G)$], which case [$a_{ij}^{[1]} = a_{ij} = 1$].
- Assume it's true whenever $1 \leq k \leq t$, and consider A_G^{t+1}. Let $i, j \in \{1, ..., n\}$.
  $a_{ij}^{[t+1]} = \sum_{l=1}^{n} a_{il}^{[t]}.a_{lj} = \sum_{l=1}^{n} N_l$, $(A_G^{t+1} = A_G^t.A_G)$, where: N_l = the number of walks $(\alpha_0 = v_i, ..., \alpha_t = v_l, \alpha_{t+1} = v_j)$ with length $t+1$ and which terminates by the edge $\{v_l, v_j\}$ (it's deduced from the hypothesis of the induction on $a_{il}^{[t]}$).
- Thus, $a_{ij}^{[t+1]}$ is the number of walks of length $(t+1)$ from v_i to v_j.

Proposition 4.4

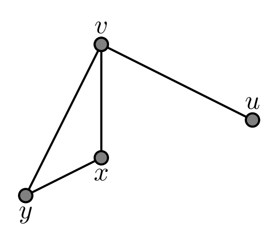
Given a graph $G = (V, E)$, if all vertices of G have degree at least two, then G contains a cycle.

Proof.

Let $P = v_0v_1...v_p$ be a longest path in G. Note that: $p \geq 2$ (because, for $x \in V$ and $y \neq z \in N(x) = \{x, y\}$ we have: yxz is a path in G). As $d(v_p) \geq 2$, there is $v \in N(v_p) \setminus \{v_{p-1}\}$.
If v is not in P (that is: if $v \notin \{v_i; 0 \leq i \leq p\}$), the path $v_0v_1...v_pv$ contradicts the choice of as the longest path.
So, there is i: $0 \leq i \leq p-2$ such that: $v = v_i$. Thus $v_iv_{i+1}...v_pv_i$ is a cycle in G.

Example 4.5

Consider the graph $G = (\{x, y, u, v\}, \{\{u, v\}, \{v, x\}, \{x, y\}, \{y, v\}\})$



G

The sequence degree of the graph G is $(1, 2, 2, 3)$, where $d_G(u) = 1 < 2$, but the graph G has a cycle $C : xyvx$.

Remarks 4.6

Let $w : v_0 = x, v_1, ..., v_p = y$ an xy-walk.

1. *We say that w connects x to y.*
2. *The vertices x and y are called the ends of the walk, x is its initial vertex and y its terminal vertex.*

3. *The vertices $v_1, ..., v_{p-1}$ are its internal vertices.*

4. *The walk w is closed if $x = y$.*

5. *A cycle of a graph G is closed trail of length ≥ 3, whose initial and internal vertices are distinct.*

5 Connected graphs

Definition 5.1

Let $G = (V, E)$ be a graph, and let u, v be two vertices in V.

1. *Two vertices u and v of G are connected if $u = v$, or if $u \neq v$ and a uv-path exists in G.*

2. *The graph G is connected if $\forall x, y \in V$, x and y are connected.*

3. *The graph G is not connected, we called G is disconnected.*
   Note *that if $|v(G)| \leq 1$, then G is connected.*

Proposition 5.2

Given a graph $G = (V, E)$, for $x, y \in V$ we denote $x\mathcal{C}y$, if x and y are connected. $\mathcal{C}$ is an equivalence relation on the set V.

Proof.

Clearly, $\mathcal{C}$ is reflexive and symmetric.
For the transitivity, consider $u, v, w \in V$, is clear if $u = v$ or $v = w$,
if not assume that: $(u \neq v, u\mathcal{C}v$, and $v \neq w, v\mathcal{C}w)$.
Let $P_1 : u_0 = u, ..., u_p = v$ be a uv-path in G, and $P_2 : v_0 = v, ..., v_q = w$ be a vw-path in G.
Then $W : u_0 = u, ..., u_p = v = v_0, ..., v_q = w$, obtained by concatenating P_1 and P_2, is an uw-walk in G. By Proposition 4.2 , we extract a uw-path in G. Thus, $u\mathcal{C}w$.

Remarks 5.3

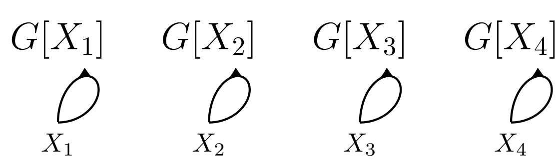
Let $G = (V, E)$ be a graph, and given $\mathcal{C}$ the equivalence relation on the set V.

1. *If X is an equivalence class of $\mathcal{C}$ on V is called connected component, and $G[X]$ is an induced subgraph of the graph G.*

2. *The graph G is connected if $\mathcal{C}$ has at most one class.*

3. *Given a disconnected graph $G = (V, E)$, then for all connected components $X \neq Y$ of G, we have: $\forall (x, y) \in X \times Y$; $\{x, y\} \notin E$. So, the connected components: $X_1, ..., X_k$ of G satisfy:*

   - *The induced subgraphs: $G[X_1], ..., G[X_k]$, are connected.*
   - *the graph G decomposed as:*



4. *A graph $G = (V, E)$ is connected if and only if $\forall x \neq y \in V$, there is an xy-path in G.*

5. *In chapter* 1*, we consider the following definition:*

   Definition *5.4*

   - *A graph* $G = (V, E)$ *is disconnected graph if* V *can be partitioned into* $\{X, Y\}$ *such that: (* $X \neq \emptyset$, $Y \neq \emptyset$, $\forall (x, y) \in X \times Y : \{x, y\} \notin E$*).*
   - *If a graph* G *is not disconnected, we say that* G *is connected graph.*
   - *Clearly, a graph* $G = (V, E)$ *is connected (in this sense) if and only if a graph* $G = (V, E)$ *is connected (in the sense of the present chapter).*

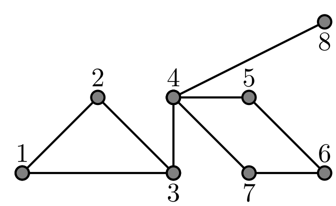
6. *Given a connected component* X *of graph* $G = (V, E)$*, we have:* $G[X]$ *is connected and for each subset* Y *of* V *such that:* $X \subset Y$*, the induced subgraph* $G[Y]$ *is disconnected.*

7. *If* $P := x_0, ..., x_p$ *is a path of* G*, then* $\{x_0, ..., x_p\}$ *is included in a connected component* X *of* G*.*

8. - *If* X *is a connected component of graph* $G = (V, E)$*, we can say that the subgraph:* $G[X]$ *is connected component of* G*.*
   - *Thus, a connected subgraph* H *of graph* G*, is a connected component if and only if* H *is not contained in any connected subgraph of* G *having more vertices or edges than* H*.*

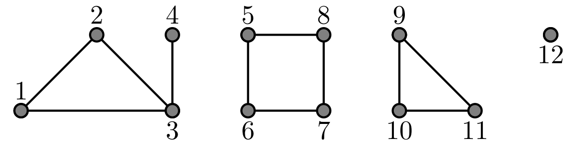
Example 5.5

1. *A graph*
   $G_1 = (\{1, 2, 3, 4, 5, 6, 7, 8\}, \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 4\}, \{4, 8\}\})$
   G_1 *is a connected graph*



G_1

2. *A graph* $G_2 = (\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\},$
   $\{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{3, 4\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 5\}, \{9, 10\}, \{10, 11\}, \{11, 9\}\})$
   G_2 *is a disconnected graph; it has exactly* 4 *connected components:*
   $X_1 = \{1, 2, 3, 4\}$, $X_2 = \{5, 6, 7, 8\}$, $X_3 = \{9, 10, 11\}$ *and* $X_4 = \{12\}$*.*



G_2

Notation 5.6 *(***Edge Cuts***)*
Let $G = (V, E)$ *be a graph and* X, Y *be two subsets of* V *(not necessarily disjoint).*

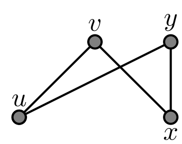
1. *$E[X,Y]$; denotes the set of edges of G with one end in X and the other end in Y.*
2. *$e(X,Y)$ denotes: $|E[X,Y]|$.*
3. *If $Y = X$, the set $E[X,X]$ is denoted $E[X]$ and $e(X,X)$ denoted $e(X)$.*
4. *If $Y = V \setminus X$, the set $E[X,Y] = E[X, V \setminus X]$ is called the* **edge cut** *of G associated with X (or the coboundary of X), and it is denoted by: $\partial(X)$.*
   *(**Note that:** $\partial(X) = E[X, V \setminus X] = \partial(V \setminus X)$).*

Remarks 5.7

1. *If $G = (V, E)$ is a graph, then $\partial(V) = \emptyset$.*
2. *A graph $G = (V, E)$ is bipartite if and only if $\partial(X) = E$ for some subset X of V.*
3. *(A graph $G = (V, E)$ is connected) if and only if $(\forall X \in \mathcal{P}(V) \setminus \{\emptyset, V\}, \partial(X) \neq \emptyset)$.*

Example 5.8

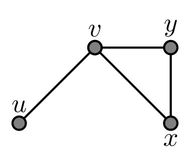
1. *A graph $G_1 = \{x, y, u, v\}, \{\{u, v\}, \{v, x\}, \{x, y\}, \{y, v\}\})$*



G_1

$\partial(\{u, v\}) = \{\{v, x\}, \{y, u\}\}$
$\partial(\{u, x\}) = \{\{u, v\}, \{u, y\}, \{v, x\}, \{x, y\}\}$
$\partial(\{u, v, y\}) = \{\{v, x\}, \{x, y\}\}$.

2. *A graph $G_2 = \{x, y, u, v\}, \{\{u, v\}, \{v, x\}, \{x, y\}, \{y, v\}\})$*



G_2

$\partial(\{u, v\}) = \{\{v, x\}, \{y, v\}\}$
$\partial(\{u, x\}) = \{\{u, v\}, \{v, x\}, \{x, y\}\}$
$\partial(\{u, v, y\}) = \{\{v, x\}, \{x, y\}\}$.

Proposition 5.9
For any graph $G = (V, E)$ and any subset X of V, we have: $|\partial(X)| = \sum_{v \in X} d_G(v) - 2e(X)$.

Proof.
Consider $s = \sum_{v \in X} d_G(v)$. In this sum, each pair $\{x, y\}$ of distinct elements of V, is:

- not counted, if $\{x, y\} \cap X = \emptyset$.
- counted once, if $|\{x, y\} \cap X| = 1$ (that is: if $\{x, y\} \in \partial(X)$).
- counted twice, if $\{x, y\} \subseteq X$ (that is: if $\{x, y\} \in E(X)$).

Thus, $s = 2|E(X)| + 1.|\partial(X)|$.

Theorem 5.10

A graph $G = (V, E)$ is even if and only if $|\partial(X)|$ is even for every subset X of V.
Recall *that G is even if: $d_G(x)$ is even for all $x \in V$.*

Proof.

- " $\Leftarrow$ " Suppose that: $\forall X \subseteq V$, $|\partial(X)|$ is even. So, $\forall v \in V$, $|\partial(\{v\})| = d_G(v)$ is even. Thus, G is even.
- " $\Rightarrow$ " Conversely, if G is even, then, give a subset X of V, we have: $\forall v \in V$, $d_G(v)$ is even and then: $\sum_{v \in X} d_G(v)$ is even, and by Proposition 5.9, $|\partial(X)| = \sum_{v \in X} d_G(v) - 2e(X)$ is even.

Proposition 5.11

Let $G = (V, E)$ be a graph of order $n \geq 1$. G is connected if and only if there is an enumeration: $u_1, ..., u_n$ of its vertices such that: $\forall k \in \{1, ..., n\}$, the induced subgraph $G[\{u_1, ..., u_k\}]$ is connected.

Proof.

- " $\Leftarrow$ " Immediate.
- " $\Rightarrow$ " Let $x \in V$ we will construct $u_1, ..., u_k$ by induction on $k \in \{1, ..., n\}$.
  Let $u_1 = x$. For $k < n$, assume that $u_1, ..., u_k$ are defined such that; $\forall i \leq k$; $G[\{u_1, ..., u_i\}]$ is connected. As $X = \{u_1, ..., u_k\} \in \mathcal{P}(V) \setminus \{\emptyset, V\}$, and G is connected, then: $\partial(X) \neq \emptyset$. So, there is $y \in V \setminus X$ and there is $i \in \{1, ..., k\}$ such that: $\{u_i, y\} \in E$. Thus, we can define u_{k+1} by: $u_{k+1} = y$. Note that: $G[\{u_1, ..., u_k, u_{k+1}\}]$ is connected, (because $\{u_1, ..., u_k, u_{k+1}\} = X \cup \{y\}$, $G[X]$ is connected and y is adjacent to an element of G (at least)).

Remarks 5.12

1. *Given a graph $G = (V, E)$ and a subset X of V such that: $G[X]$ is connected, then: $\forall y \in V \setminus X$, we have: ($G[X \cup \{y\}]$ is connected) if and only if (y is adjacent to at least, an element of X).*
2. *In the proof of Proposition 5.11, we proved that if $G = (V, E)$ is a connected graph, then: for each vertex x of G, there is an enumeration: $u_1 = x, ..., u_n$ of its vertices such that: $\forall k \in \{1, ..., n\}$, the induced subgraph $G[\{u_1, ..., u_k\}]$ is connected.*

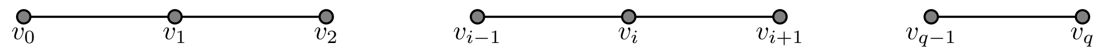
Proposition 5.13

Let $G = (V, E)$ be a connected graph of order $p \geq 2$ such that: $\forall x \in V, d(x) \leq 2$. Then G is a path P_p or a cycle C_p.

Proof.

Let $P = v_0, ..., v_q$ a longest path in G (Note that: $q \geq 1$, and P exists because G is connected).

1. If $V \neq \{v_0, ..., v_q\}$. As G is connected then: $\partial(\{v_0, ..., v_q\}) \neq \emptyset$. So, there is $\alpha \in V \setminus \{v_0, ..., v_q\}$ and there is $0 \leq i \leq q$ such that: $\{\alpha, v_i\} \in E$.
   - If $i = 0$ (resp. $i = q$) then: $P' = \alpha, v_0, ..., v_q$ is a path; contradiction: $(l(P') > l(P))$. (resp. $P' = v_0, ..., v_q, \alpha$ is a path; contradiction: $(l(P') > l(P))$.)
   - If $0 < i < q$; then: $d(v_i) \geq 3$ contradiction.
2. So, $V = \{v_0, ..., v_q\}$.



P_q

As: $\forall i;\ 0 < i < q$: $\{v_{i-1}, v_{i+1}\} \subseteq N_G(v_i)$ and $d(x) \leq 2$ for all $x \in V$, then $\{v_{i-1}, v_{i+1}\} = N_G(v_i)$.
Thus: there are two cases:

- $\{v_0, v_q\} \in E$: then G is a cycle C_q.
- $\{v_0, v_q\} \notin E$: then G is a path P_q.

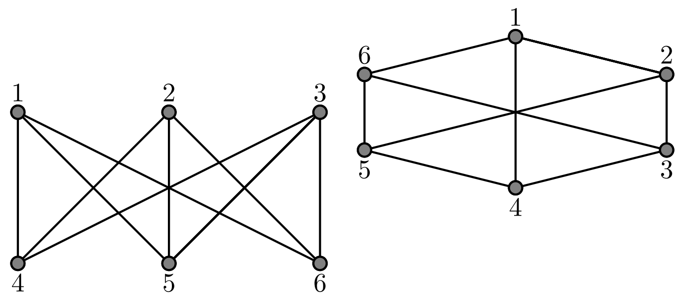
6 Isomorphic Graph

6.1 Definitions

1. - An *isomorphism* from graph $G = (V(G), E(G))$ onto a graph $H = (V(H), E(H))$ is a bijection $f : V(G) \to V(H)$ such that: $\forall x, y \in V(G), (xy \in E(G) \Leftrightarrow f(x)f(y) \in E(H))$.
   - We say that G is *isomorphic* to H (or G and H are isomorphic), and we denoted $G \simeq H$ (or $G \approx H$ or $G \cong H$), if there exists an isomorphism from G onto H.
2. - An isomorphism from a graph G onto G itself is called: an *automorphism of* G.
   - The set of automorphisms of G is denoted: $Aut(G)$.
3. - The *complement of a graph* $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ where, $\overline{E} = [V]^2 \setminus E$ (So, $\forall x \neq y \in V,\ (xy \in \overline{E} \Leftrightarrow xy \notin E)$).
   - A graph G is called *self-complementary* if it is isomorphic to its complement $\overline{G}$.

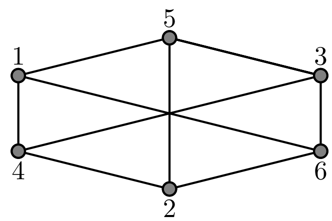
6.2 Examples

1. Let $G_1 = (\{1, 2, 3, 4, 5, 6\}, \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\})$ and let $G_2 = (\{1, 2, 3, 4, 5, 6\}, \{\{1, 2\}, \{1, 4\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 6\}, \{4, 5\}\{5, 6\}\})$



G_1 G_2

The drawing of G_1 can be transformed into the following G_2 by first moving vertex 2 to the bottom of the diagram, and the moving 5 to the top, we obtained the diagram of the graph G_1 as follows:



G_1

So, $f = \binom{1\ 2\ 3\ 4\ 5\ 6}{6\ 4\ 2\ 5\ 1\ 3}$ is an isomorphism from G_1 onto G_2.

2. Let $G_3 = (\{x, y, z, u, v, w\}, \{xy, xz, yz, uv, uw, vw, xu, yv, zw\})$

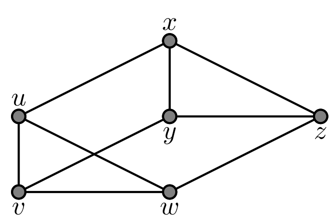
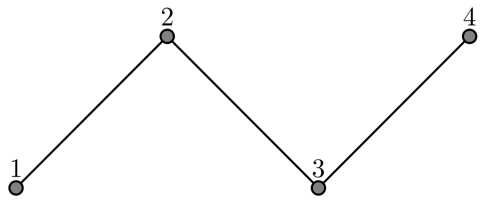
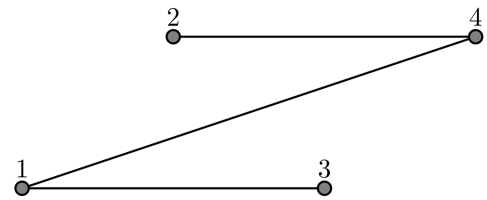


Diagram of G_3 is *the graph of prism.*
G_3 is not isomorphic to G_2 ($G_3 \ncong G_2$).

3. • Let $H_1 = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}\})$ be a graph,
$\overline{H_1} = (\{1, 2, 3, 4\}, \{\{1, 3\}, \{1, 4\}, \{2, 4\}\})$

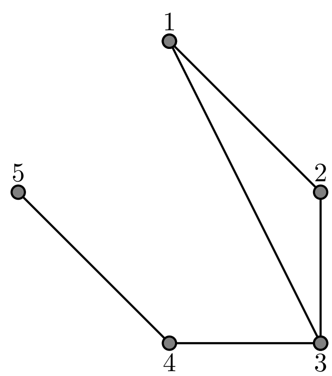


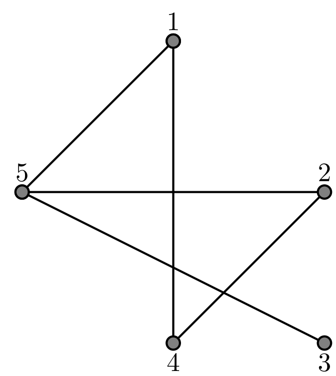


H_1 $\overline{H_1}$

The graph H_1 is self-complementary graph.

- Let $H_2 = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ be a graph, $\overline{H_2} = (\{1, 2, 3, 4, 5\}, \{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$





H_2 $\overline{H_2}$

The graph H_2 is not self-complementary graph.

6.3 Remarks

1. The relation " is isomorphic to" is an equivalence relation on the class of all graphs.

   Indeed:

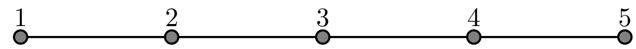
   - Let $G = (V, E)$ be a graph, we have id_V is an isomorphism from G onto G.
   - Let G and G' be two graphs, if f is an isomorphism from G onto G', the inverse mapping f^{-1} is an isomorphism from G' onto G.
   - The composite mapping, $f_2 \circ f_1$, of two isomorphisms is an isomorphism.

2. Given an isomorphism f from $G = (V, E)$ onto $G' = (V', E')$, then:

   - $\forall x \in V$, $f(N_G(x)) = N_{G'}f(x)$; so $d_{G'}(f(x)) = d_G(x)$.
   - $|V| = |V'|$, $|E| = |E'|$, and G and G' have the same degree sequence.

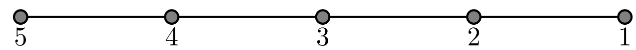
3. Let $G(V,E)$ be a graph. $(Aut(G), \circ)$ is a group (it is a subgroup of $(S_V, \circ)$ the group of permutations of V). The group $(Aut(G)$ is called the *automorphism group of* G.

   Example
   Let $G = (\{1,2,3,4,5\}, \{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\}$ be a graph.



G

   The graph G is too $G = (\{1,2,3,4,5\}, \{\{5,4\},\{4,3\},\{3,2\},\{2,1\}\}$.



G

   $f = \binom{1\,2\,3\,4\,5}{5\,4\,3\,2\,1}$ is an automorphism.

6.4 Properties

Proposition 6.1
If a graph G is self-complementary, then his order n satisfies: $n \equiv 0\ (mod\ 4)$ or $n \equiv 1\ (mod\ 4)$ (i. e $n = 4p$ or $n = 4p+1$, where $p \in \mathbb{N}$).

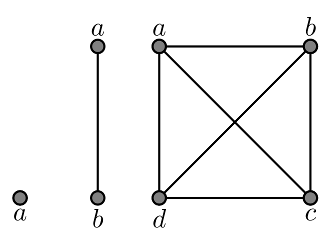
Proof.
Let $G = (V,E)$ be a self-complementary graph, then $|E| = |[V]^2 \setminus E|$, hence $2|E| = |[V]^2| = \binom{n}{2} = \frac{n(n-1)}{2}$, therefore $|E| = \frac{n(n-1)}{4}$, since $|E| \in \mathbb{N}$, hence 4 devises $n(n-1)$, therefore $n \equiv 0\ (mod\ 4)$ or $n \equiv 1\ (mod\ 4)$.

7 Particular Graphs

7.1 Complete Graph

- A *complete graph* is a graph in which any two vertices (different vertices) are adjacent.
- Up to isomorphy, for each integer $n \geq 1$, there is a unique complete graph of order n. It is denoted: K_n.

Examples:



K_1 K_2 K_4

7.2 Empty Graph

- An *empty graph* is a graph $G = (V, E)$ with: $E = \emptyset$.
- Up to isomorphy, for each integer $n \geq 1$, there is a unique empty graph of order n. It is denoted: D_n.

Examples:

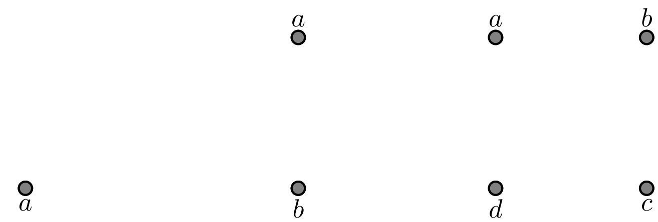


Diagram of D_1 Diagram of D_2 Diagram of D_4

7.3 Paths

A *path* is a graph isomorphic to the graph: $P_n = (\{1, ..., n\}, \{\{i, i+1\};\ 1 \leq i \leq n-1\})$.

Examples:

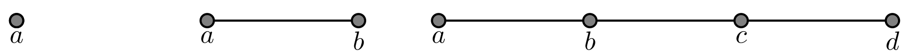


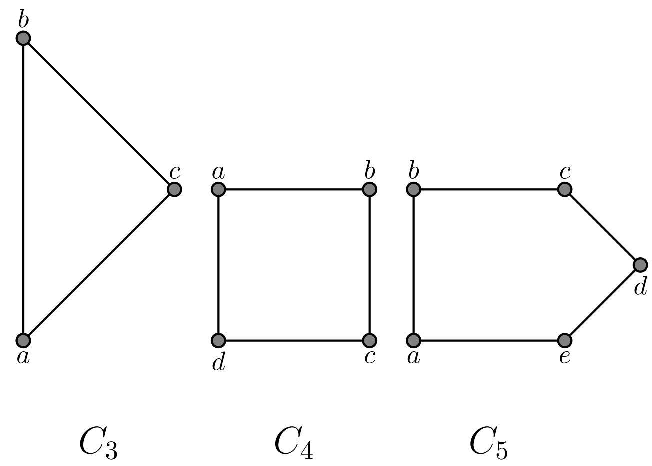
Diagram of P_1 Diagram of P_2 Diagram of P_4

7.4 Cycles

1. A *cycle* on $n \geq 3$ is a graph isomorphic to the graph:
   $C_n = (\{1, ..., n\}, \{\{i, i+1\};\ 1 \leq i \leq n-1\} \cup \{\{1, n\}\})$.
2. - The *length* of a path or a cycle is the number of its edges.
   - *k-path* (resp. *k-cycle*) is a path (resp. cycle) of length k.
   - A *k-path* (resp. *k-cycle*) is *odd* or *even* according to the parity of length k.
   - A 3-cycle is often called a *triangle*.

Remark 7.1
The cycle C_n is obtained from the path P_n by adding the edge $\{1, n\}$.

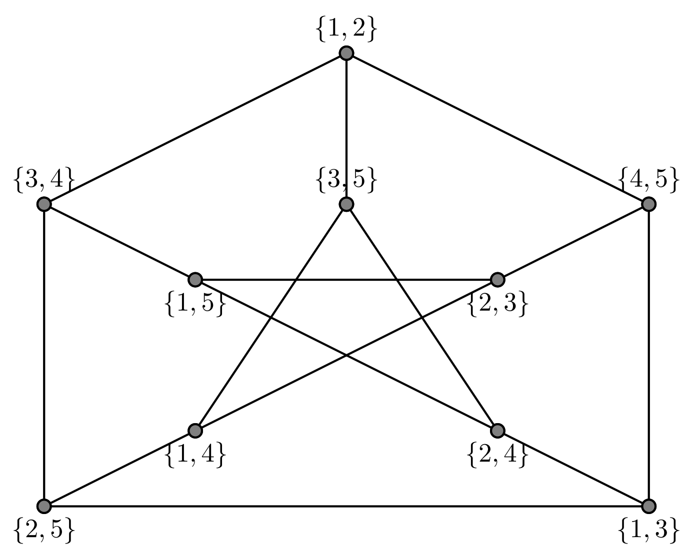
Examples:



7.5 Petersen graph

A *Petersen graph* is a graph $G = (V, E)$, up to isomorphy, defined by:
$V = \mathcal{P}_2(\{1,2,3,4,5\}) = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$
and for $i \neq j \in \{1,2,3,4,5\}$ and $\alpha \neq \beta \in \{1,2,3,4,5\}$ where:
$(\{\{i,j\},\{\alpha,\beta\}\} \in E) \Leftrightarrow (\{i,j\} \cap \{\alpha,\beta\} = \emptyset)$

Diagram of Petersen graph

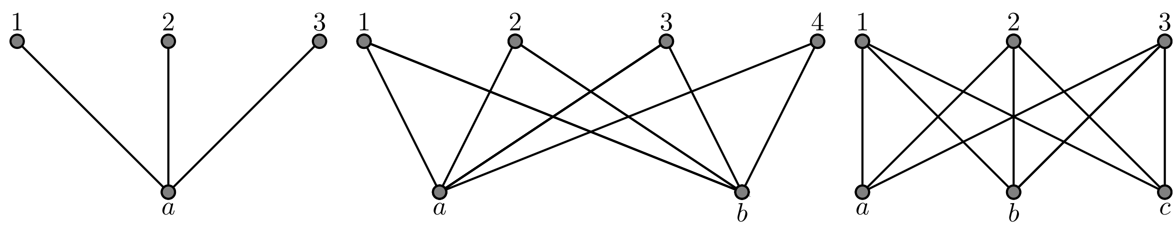


Petersen graph

7.6 Bipartite graphs

1.
   - A *Bipartite graph* is a graph $G = (V, E)$, such that V can be partitioned into two subsets X and Y such that every edge has one end in X and one in Y.
   - Such a partitioned $\{X, Y\}$ is called *a partition of the graph* G; X and Y are the parts of V, in this case G is denoted: $G[X, Y]$.
2.
   - If $G[X, Y]$ is a bipartite graph such that every $x \in X$ is joined to every $y \in Y$, then G is called a *Complete bipartite graph.*
   - Up to isomorphy, we denoted $K_{p,q}$ the complete bipartite graph $G[X, Y]$ with: $|X| = p$ and $|Y| = q$.

Examples:

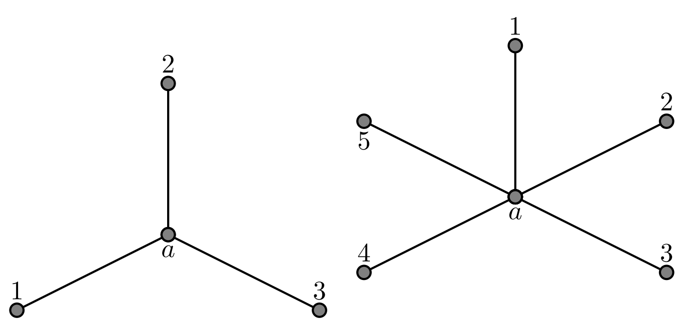


$K_{1,3}$ $K_{2,4}$ $K_{3,3}$

7.7 Star graphs

A *Star* is a complete bipartite graph $G[X, Y]$ with: ($|X| = 1$ or $|Y| = 1$).

Examples:

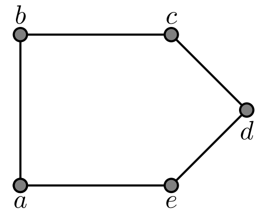


$K_{1,3}$ $K_{5,1}$

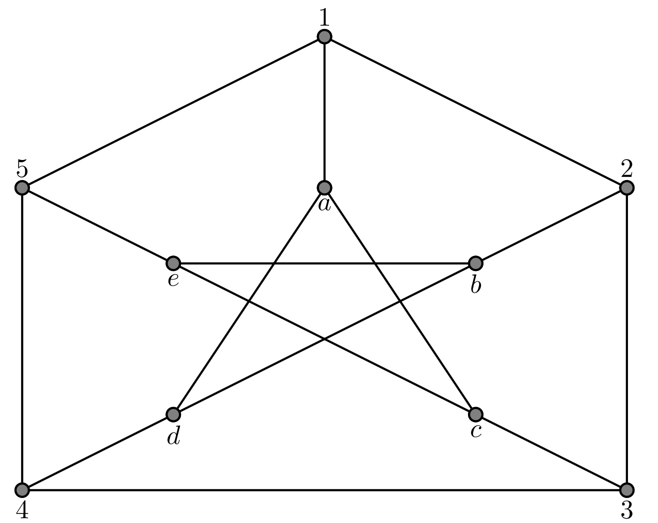
7.8 Regular graphs

- A *k-regular graph*, where $k \in \mathbb{N}$ is a graph $G = (V, E)$, such that: $\forall x \in V,\ d(x) = k$.
- A *regular graph* is a graph which is k-regular graph for some k.

Examples:



C_5 is a 2-regular graph



A Patersen graph is 3-regular graph

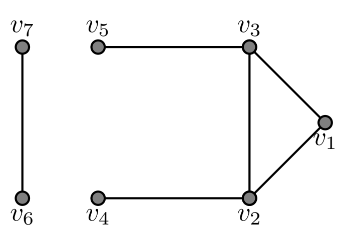
In general, C_n is a 2-regular graph.

7.9 Disconnected graphs

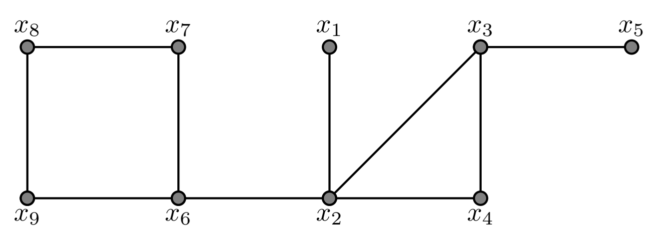
- A *disconnected graph* is a graph $G = (V, E)$, where V can be partitioned into $\{X, Y\}$ such that:$(X \neq \emptyset,\ Y \neq \emptyset,\ \forall (x, y) \in X \times Y : \{x, y\} \notin E)$.
- If a graph G is not disconnected, we say that G is *connected graph*.

Examples:

- Given a graph $G = (\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \{v_1v_2, v_2v_3, v_3v_1, v_2v_4, v_3v_5, v_6v_7\})$.
- Given a graph $H = (\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\},$ $\{x_1x_2, x_2x_3, x_3x_4, x_2x_4, x_2x_6, x_3x_5, x_6x_7, x_7x_8, x_8x_9, x_6x_9\})$.



G is disconnected graph



H is connected graph

8 Degree Sequence

8.1 Definition

Definition 8.1

We say that an increasing sequence $D = (d_1, ..., d_n)$ is graphic if there is a graph G having D as the degree sequence (i.e: $D = DEG(G)$).

Remarks 8.2

If an increasing sequence $D = (d_1, ..., d_n)$ is graphic, then

1. $d_n \leq n - 1$.
2. *D has an even odd terms.*
3. *If* $d_1 = 0$, *then* $d_n \leq n - 2$.
   If $d_n = n - 1$, *then* $d_1 \geq 1$.
4. *We remark there are* $i \neq j$ *such that* $d_i = d_j$.

Notation 8.3

1. $D = (d_1, ..., d_n)$ *an increasing sequence of integers with:*
   $0 < d_1 \leq ... \leq d_n < n$, *where* $n \geq 2$.
2. $D'' = (d''_1, ..., d''_{n-1})$ *the sequence obtaining, from* D, *as follows:*
   - *delete* d_n *from* D *and*
   - *Subtract* 1 *from each of the* d_n *last remaining terms.*
3. $D' = (d'_1, ..., d'_{n-1})$ *the increasing sequence consists of integers* $\{d''_1, ..., d''_{n-1}\}$ *arranged in ascending order.*

8.2 Havel-Hakimi Theorem

Problematic:

A degree sequence can be obtained from graph. But how to get graph from degree sequence? There can be many graph from a degree sequence or there can not be any graph.
So, how to know if a degree sequence is a **graphic** sequence?
The solution is the Havel-Hakimi Theorem.

Theorem 8.4 Havel-Hakimi Theorem
The sequence D is graphic if and only if the sequence D' is graphic.

Consequence

1. This theorem reduces the study of D to the study D'.
2. Thus, we have an algorithmic test to check whether D is graphic and to generate a graph whenever one exists.

We remark, this theorem is easily deduced from the following lemma.

Lemma 8.5

Let $D = (d_1, ..., d_n)$ be a graphic sequence with: $d_n > 0$ (and then $n \geq 2$). Then there is a graph $G = (V, E)$ where, $V = \{x_1, ..., x_n\}$, such that:

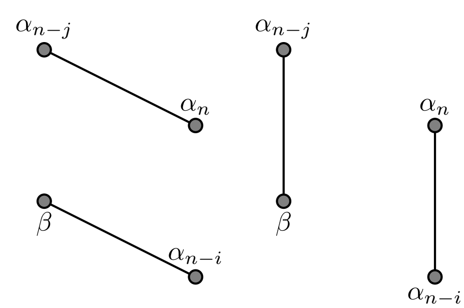
- $\forall i \in \{1, ..., n\}$, $d_G(x_i) = d_i$, *and*

- $N_G(x_n) = \{x_{n-i};\ 1 \leq i \leq d_n\}$

Proof.

By contradiction.

1. Consider a graph $G = (\{\alpha_1, ..., \alpha_n\}, E)$ with:
   (a) $\forall i, d_G(\alpha_i) = d_i$
   (b) The cardinality $|N_G(\alpha_n) \cap \{\alpha_{n-i}; 1 \leq i \leq d_n\}|$ is **maximum** (for all graphs G with: $DEG(G) = D$).
2. So,
   (a) $\exists i, 1 \leq i \leq d_n, \{\alpha_{n-i}, \alpha_n\} \notin E$
   (b) $\exists j, d_n + 1 \leq j \leq n-1, \{\alpha_{n-j}, \alpha_n\} \in E$
   (c) We may assume that: $d_{n-j} < d_{n-i}$
   (d) As $\alpha_n \in N_G(\alpha_{n-j}) \setminus N_G(\alpha_{n-i})$, there are $\beta \neq \lambda \in N_G(\alpha_{n-i}) \setminus N_G(\alpha_{n-j})$, (with $\beta \neq \alpha_{n-j}$).
3. Thus, β, α_{n-j}, α_{n-i} and α_n are 4 distinct vertices of G, with: $\{\beta, \alpha_{n-i}\} \in E(G)$ and $\{\alpha_{n-j}, \alpha_n\} \in E(G)$; and $\{\beta, \alpha_n\}$ is an edge or not.
4. We consider the graph G' such that, β, α_{n-j}, α_{n-i} and α_n are 4 distinct vertices verifies of G', with $\{\beta, \alpha_{n-j}\} \in E(G')$ and $\{\alpha_{n-i}, \alpha_n\} \in E(G')$, the other edges are the same on G, hence, $DEG(G) = DEG(G') = D$, is a **contradiction** by the cardinality $|N_G(\alpha_n) \cap \{\alpha_{n-i}; 1 \leq i \leq d_n\}|$ is maximum.



$G[\{\beta, \alpha_{n-j}, \alpha_{n-i}, \alpha_n\}]$ $G'[\{\beta, \alpha_{n-j}, \alpha_{n-i}, \alpha_n\}]$

Algorithm of Havel-Hakimi

1. **Step** 1
   Sort the sequence in **increasing sequence** $D = (d_1, ..., d_n)$
2. **Step** 2
   - Remove the term d_n in a sequence D.
   - Subtract 1 from each the d_n last terms in the sequence $(d_1, ..., d_{n-1})$.
3. **Step** 3

- If a negative number in this new sequence, we stopped and the sequence $D = (d_1, ..., d_n)$ is not graphic.
- If all number zeros in this new sequence, we stopped and the sequence $D = (d_1, ..., d_n)$ is graphic.
- Otherwise, we arranged in ascending order this new sequence, consider $D' = (d_1', ..., d_{n-1}')$ the new increasing sequence obtained, and repeat from step 1.

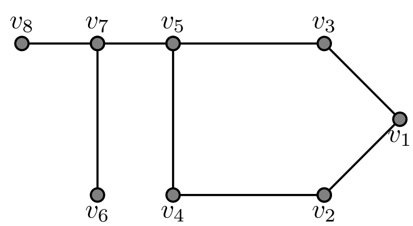
Example 8.6

Prove that the sequence $(1, 2, 3, 5, 3, 1, 2, 3)$ *is a graphic sequence and give an example of a graph* G *satisfying* $DEG(G) = D$*.*

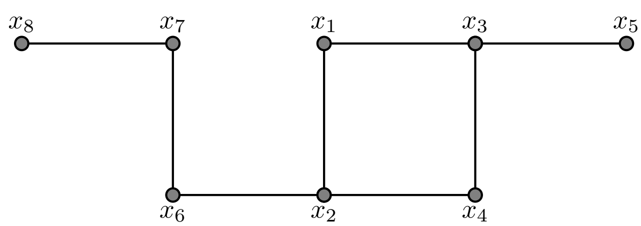
Remark 8.7 *For the same degree sequence that is graphic, it is possible to find more than one graph which are not isomorphic.*

Example 8.8

Given a graph $G = (\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, \{v_1v_2, v_3v_1, v_2v_4, v_3v_5, , v_4v_5, v_5v_7, v_6v_7, v_7v_8\})$*, and a graph* $H = (\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \{x_1x_2, x_1x_3, x_3x_4, x_2x_4, x_2x_6, x_3x_5, x_6x_7, x_7x_8\})$*.*



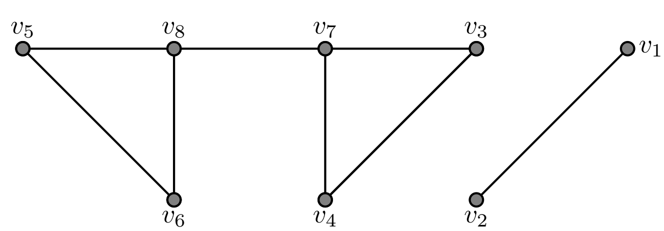
Graph G



Graph H

The two graphs G and H are not isomorphic, and they have the same degree sequence $(1, 1, 2, 2, 2, 2, 3, 3)$*.*

Using the Havel-Hakimi algorithm for the same degree sequence $(1, 1, 2, 2, 2, 2, 3, 3)$*, we find the graph* K *as follows:*



Graph K

The graphs K *and* G *are not isomorphic, and the graphs* K *and* H *are not isomorphic.*

Exercise 8.9

1. *Is the increasing sequence* $D = (d_1, ..., d_n)$ *a* **graphic** *sequence?*

(a) $D = (1, 2, 3, 4, 4, 5, 6, 7)$

(b) $D = (2, 2, 3, 3, 6, 6, 6, 6)$

(c) $D = (2, 2, 3, 4, 4, 6, 6, 7)$

(d) $D = (1, 1, 2, 4, 6, 7, 7, 8)$

2. *In the case, where D is graphic, give an example of graph G satisfying $DEG(G) = D$.*

9 Exercises of Graphs

Exercise 9.1

1. *How many edges does a graph with a sum of degrees of vertices of 48 have?*
2. *How many edges does a regular graph of type 2 with 14 vertices have?*
3. *A graph with 47 edges, what is the minimum possible number of vertices?*
4. *Does a graph with vertex degrees 3,2,2,2,2 exist?*
5. *Is the adjacency relation between vertices in graphs transitive?*
6. *Give an example of a connected graph that does not contain cycles and has four vertices.*

Exercise 9.2

1. *If $\delta(G) \geq 2$, prove that G contains a cycle*
2. *Give an example of a graph with five vertices and two components.*
3. *Give an example of a graph with five vertices, each of degree two.*

Exercise 9.3

1. *Prove that the number of edges of a regular graph of type r with n vertices is $\frac{nr}{2}$.*
2. *How many edges does a complement of bipartite complete graph $\overline{K_{m,n}}$ have?*
3. *Let $G = (V, E)$ be a graph such that $|V| = n$ and $|E| \geq \frac{n^2}{4} + 1$. Prove that G cannot be bipartite.*
4. *If G is a graph that does not contain odd cycles, use mathematical induction on the number of edges to prove that G is a bipartite graph.*
5. *What is the minimum number of edges that must be removed from K_n to make it disconnected?*
6. *Prove that if there is a path between two vertices in a graph, then there is a path of length at most $n - 1$ between them.*
7. *Prove that the graph isomorphism relation is an equivalence relation on the set of all graphs.*

Exercise 9.4

1. *Show that: a graph is bipartite if and only if it contains no odd cycle.*
2. *Let $G = (V, E)$ be a graph. Show that if $\delta(G) \geq 2$, then G contains a cycle.*

Exercise 9.5

1. (a) *Is the increasing sequence $D = (d_1, ..., d_n)$ a* **graphic** *sequence?*
   i. $D = (1, 2, 3, 4, 4, 5, 6, 7)$

ii. $D = (2, 2, 3, 3, 6, 6, 6, 6)$
iii. $D = (2, 2, 3, 4, 4, 6, 6, 7)$
iv. $D = (1, 1, 2, 4, 6, 7, 7, 8)$

(b) *In the case, where D is graphic, give an example of graph G satisfying $DEG(G) = D$.*

2. *Consider the two sequences:* $D_1 = (1, 1, 3, 3, 3, 3, 5, 6, 8, 9)$ *and* $D_2 = (3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6, 6, 6)$
   (a) *Show that: D_1 is not graphic.*
   (b) *Show that: D_2 is graphic and give a graph G with $DEG(G) = D_2$.*
   (c) *Show that there is no bipartite graph G such that $DEG(G) = D_2$.*

Exercise 9.6

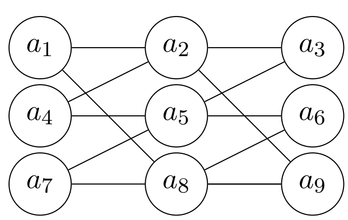
1. *A graph G is called a **self-complementary** graph if $G \cong \bar{G}$. Give an example of a self-complementary graph with four vertices and another with five vertices.*
2. *If G is a self-complementary graph with n vertices, prove that either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.*

Exercise 9.7

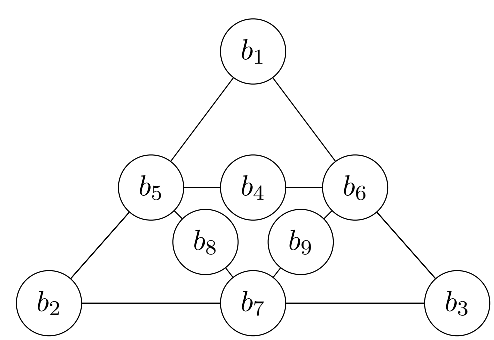
1. *Given a graph G with six vertices, prove that either G or its complement $\bar{G}$ contains a complete subgraph (K_3 or a triangle).*
2. *Give an example that shows that the result in Question 1. is not true when the number of vertices is five.*
3. *Prove that if there are only two vertices of odd degree in a graph, then they belong to the same component.*

Exercise 9.8

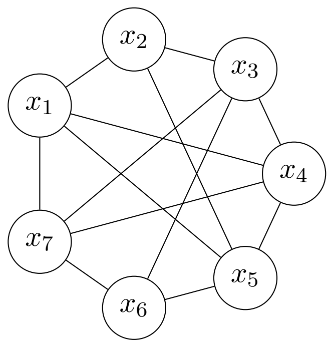
Consider the following graphs G_1, G_2, H_1, H_2.



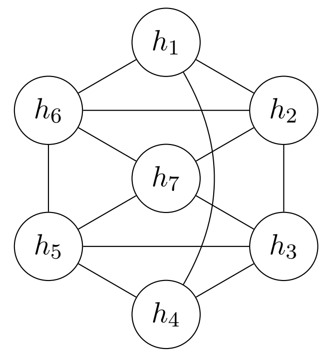
Graph G_1



Graph G_2



Graph H_1



Graph H_2

1. *Give the order and size of each graph.*
2. *Determine the degree sequence of each graph.*
3. *Which graphs are bipartite? Justify.*
4. *Which graphs are planar?*
5. *Are there isomorphic graphs among these figures?*

Exercise 9.9

Find all graph self complementary, up to isomorphy, of order 5 *or less.*

Exercise 9.10

1. *Find the adjacency matrix of the graph* H_1 *given in Exercise 6.8.*
2. *Draw the graph* $G = (\{a, b, c, d, f\}, E)$ *whose adjacency matrix is:*

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

3. *Find the number of paths of length 4 from vertex* a *to vertex* d *in the graph* G*¿*

Exercise 9.11

1. *Does a bipartite graph exist such that the sequence* $(9, 8, 6, 5, 3, 3, 3, 3, 3, 1, 1)$ *is a degree sequence for it?*
2. *Does a bipartite graph exist such that the sequence* $(6, 6, 6, 6, 6, 5, 3, 3, 3, 3, 3)$ *is a degree sequence for it?*
3. *In the following, find a realization (drawing) for the given degree set:*
   (a) $D = \{4, 3, 2, 1\}$
   (b) $D = \{3, 4, 5, 7\}$