

# Chapter One

## Basic Counting Principles

Counting is a fundamental goal in the study of **Combinatorics**. Combinatorics is sometimes called the "art of counting" because it allows us to count the elements of a finite set without listing its elements in a detailed list.

This chapter aims to introduce the six standard types of counting. These six problems, in addition to the principles of **Sum**, **Product**, and **Bijection**, represent the primary tools for solving most of the problems in this course.

Four of these six problems—**Sequences**, **Permutations**, **Combinations**, and **Multisets**—can be viewed through two models : the *Sampling Model* and the *Distribution Model*. Meanwhile, the remaining two problems—**Set Partitioning** and **Integer Partitioning**—can only be viewed through the *Distribution Model*.

### 1.1 The Rules of Sum and Product

#### The Rule of Sum

If  $A_1, A_2, \dots, A_k$  are finite sets such that  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ , then :

$$|A_1 \cup A_2 \cup \dots \cup A_k| = |A_1| + |A_2| + \dots + |A_k|$$

This is called **The Rule of Sum**. The proof by mathematical induction on  $k$  is left to the reader.

**Equivalent Formulation :** If completing task  $A$  requires performing any one of the tasks  $\{A_1, A_2, \dots, A_k\}$ , and no two tasks  $A_i$  and  $A_j$  can be performed simultaneously, then the number of ways to complete  $A$  is :

$$n_1 + n_2 + \dots + n_k$$

where  $n_i$  is the number of ways to perform task  $A_i$ .

#### The Rule of Product

If  $A_1, A_2, \dots, A_k$  are finite sets, then :

$$|A_1 \times A_2 \times \dots \times A_k| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_k|$$

This is called **The Rule of Product**.

**Equivalent Formulation :** If completing task  $A$  requires completing a sequence of tasks  $A_1, A_2, \dots, A_k$  (first  $A_1$ , then  $A_2$ , and so on), and the number of ways to complete task  $A_i$  does not depend on how the previous tasks were completed, then the total number of ways to complete  $A$  is :

$$n_1 \cdot n_2 \cdot \dots \cdot n_k$$

### Example (1.1)

Let  $\Sigma$  be an alphabet with  $m$  letters. Find  $|\Sigma_n|$ , where  $\Sigma_n$  is the set of all words of length  $n$  whose letters are taken from the alphabet  $\Sigma$ .

**Solution :** Let  $w = x_1x_2 \dots x_n$  be a word from  $\Sigma_n$ . The number of ways to choose each letter  $x_i$  is  $m$  for all  $1 \leq i \leq n$ . Since the choice of letter  $x_i$  does not depend on the choice of the letters before it, based on the **Rule of Product** :

$$|\Sigma_n| = m^n$$

### Example (1.2)

A hospital employs 4 doctors, 7 nurses, and 3 technicians. In how many ways can a task force be formed consisting of one doctor, one nurse, and one technician ?

**Solution :** A doctor can be chosen in 4 ways, a nurse in 7 ways, and a technician in 3 ways. According to the **Rule of Product**, the number of possible ways is :

$$4 \cdot 7 \cdot 3 = 84$$

### Example (1.3)

How many two-digit numbers can be formed such that the sum of their digits is an odd number ?

**Solution :** Let  $y$  be the units digit and  $x$  be the tens digit. We begin by choosing  $x$ .  $x$  can be chosen from the set  $\{1, 2, \dots, 9\}$ , so there are 9 ways to choose  $x$ . If  $x$  is odd, then  $y$  can be chosen from the set  $\{0, 2, 4, 6, 8\}$ . If  $x$  is even, then  $y$  can be chosen from the set  $\{1, 3, 5, 7, 9\}$ . In either case, the number of ways to choose  $y$  after choosing  $x$  is 5. Thus, the number of required integers is  $9 \cdot 5 = 45$ . Note that if we had started by choosing  $y$ , the number of ways to choose  $x$  after  $y$  would not be constant.

## 1.2 The Bijection Principle

If  $f : A \rightarrow B$  is a bijection from set  $A$  to set  $B$ , then  $|A| = |B|$ .

## 1.3 The Sampling Model for Counting

Let  $A = \{a_1, a_2, \dots, a_n\}$  be a set. Taking a sample from  $A$  depends on the answers to the following two questions :

1. Is the **order** of elements important in this sample?
2. Is **repetition** of an element in the sample allowed?

Therefore, we have four cases, which the following example will illustrate.

### Example (1.4)

Let  $A = \{a_1, a_2, a_3\}$ . The following table illustrates the samples of size two taken from  $A$ :

	Repetition Allowed	Repetition Not Allowed
Order Important	$(a_1, a_1), (a_1, a_2), (a_1, a_3),$ $(a_2, a_1), (a_2, a_2), (a_2, a_3),$ $(a_3, a_1), (a_3, a_2), (a_3, a_3)$	$(a_1, a_2), (a_1, a_3), (a_2, a_1),$ $(a_2, a_3), (a_3, a_1), (a_3, a_2)$
Order Not Important	$\{a_1, a_1\}, \{a_1, a_2\}, \{a_1, a_3\},$ $\{a_2, a_2\}, \{a_2, a_3\}, \{a_3, a_3\}$	$\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}$

**Sequences :** Let  $A = \{a_1, a_2, \dots, a_n\}$ . We call the sample an  **$r$ -sequence** from  $A$  if repetition is allowed and the order is important. It is written in the form  $(x_1, x_2, \dots, x_r)$ . Similar to Example (1.1), the number of  $r$ -sequences from a set of size  $n$  is  $n^r$ .

**Permutations :** Let  $A = \{a_1, a_2, \dots, a_n\}$ . We call the sample an  **$r$ -permutation** from  $A$  if repetition is not allowed and the order is important. It is written in the form  $(x_1, x_2, \dots, x_r)$ .

### Theorem (1.1)

Let  $p, n$  be two non negative integers.

If  $1 \leq p \leq n$ . Then, the number of  $r$ -permutations from a set of  $n$  elements is

$$n(n-1)\dots(n-r+1)$$

**Proof:** Let  $A = \{a_1, a_2, \dots, a_n\}$  be a set and let  $P = (x_1, x_2, \dots, x_r)$  be an  $r$ -permutation from  $A$ . Note that :

1. The number of ways to choose  $x_1$  is  $n$ .
2. The number of ways to choose  $x_2$  is  $n-1$ .
3. The number of ways to choose  $x_3$  is  $n-2$ .
- $\vdots$
- $r$ . The number of ways to choose  $x_r$  is  $n-r+1$ .

Since the choices are independent at each stage, according to the **Rule of Product**, the number of  $r$ -permutations from  $A$  is  $n(n-1)\dots(n-r+1)$ . ■

We usually denote the number of  $r$ -permutations from a set of size  $n$  by the symbol  $(n)_r$  or by  $P(n, r)$ . Note that :

$$(n)_r = P(n, r) = \frac{n!}{(n-r)!} = n(n-1)\dots(n-r+1)$$

Let  $A = \{a_1, a_2, \dots, a_n\}$ . Any **subset** of  $A$  with size  $r$  can be considered a sample of size  $r$  from  $A$  where the order is not important and repetition is not allowed. A subset is sometimes called a **combination**.

### Theorem (1.2)

If  $0 \leq r \leq n$ , then the number of subsets of size  $r$  from a set of  $n$  elements is

$$\frac{n!}{r!(n-r)!}$$

**Proof :** If  $r = 0$ , then the empty set is the only subset, which contains no elements.

Suppose  $r > 0$ . Note that any subset with  $r$  elements corresponds to  $r!$  different permutations in the set of  $r$ -length permutations. Likewise, we can obtain  $r!$  different permutations from any subset of size  $r$ . Therefore :

$$r! \times (\text{number of subsets of size } r) = \text{number of permutations of length } r$$

From Theorem (1.1), we conclude that the number of subsets with  $r$  elements is equal to :

$$\frac{n!}{r!(n-r)!} \blacksquare$$

The number of subsets of size  $r$  from a set of size  $n$  is denoted by the symbol  $\binom{n}{r}$  or  $C(n, r)$ .

### Example (1.5)

If an exam paper contains 7 questions and a student has to answer only 5 questions, in how many ways can the student answer the exam ?

**Solution :** The number of ways to answer is :

$$\binom{7}{5} = 21$$

### Example (1.6)

There are 12 engineers working in a company. To carry out a project, the company wants to select a work team consisting of 5 engineers.

- In how many ways can the company choose the work team ?
- In how many ways can the company choose the work team if two engineers insist on working together ?
- In how many ways can the company choose the work team if two engineers refuse to work together ?

**Solution (1.6) :** (a) The number of possible ways is :  $\binom{12}{5} = 792$ .

(b) Let  $a$  and  $b$  be the engineers who insist on working together. If  $a$  and  $b$  are part of the selected team, then the number of ways to complete the team is  $\binom{10}{3}$ . If the selected team includes neither  $a$  nor  $b$ , then the number of ways to choose the team is  $\binom{10}{5}$ . Therefore, based on the **Rule of Sum** :

$$\binom{10}{3} + \binom{10}{5} = 120 + 252 = 372$$

(c) Let  $a$  and  $b$  be the engineers who refuse to work together. If  $a$  is in the team, then  $b$  is not, and the number of ways is  $\binom{10}{4}$ . Similarly, if  $b$  is in the team, then  $a$  is not, and the number of ways is  $\binom{10}{4}$ . If the team contains neither  $a$  nor  $b$ , the number of ways is  $\binom{10}{5}$ . Thus, based on the **Rule of Sum** :

$$\binom{10}{4} + \binom{10}{4} + \binom{10}{5} = 210 + 210 + 252 = 672$$

**Multisets :** Let  $A = \{a_1, a_2, \dots, a_n\}$ . We call the sample an  $r$ -**multiset** from  $A$  if the order is not important and repetition is allowed. It is written in the form  $M = \{2 * a, b, 3 * c, d\}$ .

According to what was mentioned above,  $M = \{a, a, b, c, c, c, d\}$  is a multiset of size 7, where the repetitions of  $a, b, c, d$  are 2, 1, 3, 1 respectively. We also write  $M$  as  $M = a, b, c$

### Example (1.7)

The multisets of size 3 from  $\{a, b, c\}$  are :

$$\{a, a, a\}, \{a, a, b\}, \{a, a, c\}, \{a, b, c\}, \{b, b, b\}, \\ \{b, b, a\}, \{b, b, c\}, \{c, c, c\}, \{c, c, a\}, \{c, c, b\}.$$

### Theorem (1.3)

The number of  $r$ -multisets from a set of size  $n$  is equal to :

$$\binom{n-1+r}{r}$$

**Proof :** Let  $A = \{a_1, a_2, \dots, a_n\}$  be a set. For each multiset of size  $r$  from  $A$ , we form a table consisting of  $n$  columns and two rows. In the first row, from left to right, we place the elements  $a_1, a_2, \dots, a_n$ . For each  $i = 1, 2, \dots, n$ , we place stars in the cell below  $a_i$  equal to the number of times  $a_i$  appears in the multiset. Note that the total number of stars in the second row is  $r$ . For example, the multiset  $\{a_1, a_1, a_1, a_3\}$  from the set  $\{a_1, a_2, a_3\}$  is represented by the following table :

$a_1$	$a_2$	$a_3$
***		*

Conversely, any table where the number of stars in the second row is  $r$  corresponds to a multiset from  $A$  of size  $r$ . Therefore, there is a bijection between multisets and these tables. To calculate the number of tables, we make the following change :

We remove the three horizontal lines, the elements from the first row, and the first and last vertical lines of the table. For example, the table above becomes : \*\*\*||\*. Conversely, \*\*|\*\*| corresponds to the multiset  $\{a_1, a_1, a_2, a_2\}$ . Thus, the number of multisets of size  $r$  is equal to the number of permutations of  $r$  stars and  $n - 1$  vertical bars. To calculate this number, we have  $n - 1 + r$  positions. If we choose  $r$  positions for the stars, the remaining positions will be for the vertical bars. Since the number of ways to choose  $r$  positions from  $n - 1 + r$  positions is :

$$\binom{n - 1 + r}{r}$$

then the number of multisets of size  $r$  is :

$$\binom{n - 1 + r}{r} \blacksquare$$

## Exercises (1, 1)

- (a) How many integers between 1 and 99 have no repeated digits?  
 (b) How many even integers between 1 and 99 have no repeated digits?  
 (c) How many odd integers between 1 and 99 have no repeated digits?
- Let  $B = \{100, 101, \dots, 999\}$ . Find the number of odd integers in  $B$  with distinct digits.
- Let  $B = \{1000, 1001, \dots, 9999\}$ . Find the number of odd integers in  $B$  with distinct digits.
- A coin is tossed 30 times. How many possible sequences of heads and tails are there?
- How many ways are there to answer 20 "Yes/No" questions?
- How many ways are there to answer a 50-question exam if the first 20 questions have 3 choices each and the remaining 30 have 5 choices each?
- How many ways can the letters of the word **COMPUTER** be arranged :  
 (a) If the vowels are adjacent?  
 (b) If the letter P appears to the left of T?  
 (c) If there are exactly two letters between M and C?
- Calculate the following :

$$\binom{10}{7}, \binom{8}{1}, \binom{5}{2} \cdot \binom{7}{3}$$

- Simplify the following expressions :

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{n-1}, \binom{n}{n}$$

10. Find the values of :  $(n)_1, (7)_6, (8)_4, (8)_3$ .
11. Show that  $\binom{n}{n-1} = \binom{n}{1}$ .
12. Show that  $\binom{n}{k} = \binom{n}{n-k}$ .
13. Prove that  $\binom{n}{k} = \frac{n}{n-k} \binom{n-1}{k}$ .
14. Using Exercise 13, prove that  $\binom{2n}{n}$  is an even number for  $n \geq 1$ .
15. Prove that for any positive integer  $n$ , there exist at least  $n$  consecutive composite (non-prime) numbers.  
*Hint : Consider the numbers :  $(n+1)! + 2, (n+1)! + 3, \dots, (n+1)! + (n+1)$ .*
16. In how many ways can  $n$  people be seated around a round table ?
17. A ship sends signals by raising seven flags sequentially on a mast. How many different signals can be sent using five flags of different colors ?
18. In how many ways can four identical red cars and four identical white cars be lined up such that no two cars of the same color are adjacent ?
19. In how many ways can four distinct red cars and four distinct white cars be lined up such that no two red cars are adjacent ?
20. How many 5-letter words can be formed from the English alphabet if no letter can be repeated ?
21. A student has 25 different books and a shelf that only fits 10 books. In how many ways can he arrange 10 of his books on the shelf ?
22. How many permutations of  $\{1, 2, \dots, n\}$  fix the number 1 ?
23. In how many ways can 12 distinct elements be partitioned into three groups of four elements each ?
24. In how many ways can  $2n$  distinct elements be partitioned into  $n$  groups of two elements each ?
25. In how many ways can  $mn$  distinct elements be partitioned into  $m$  groups of  $n$  elements each ?
26. Prove that  $r!$  divides the product of any  $r$  consecutive positive integers.  
*Hint : Consider the number of ways to choose  $r$  elements from a set of  $n+r-1$  elements.*
27.  $n$  different sticks are each broken into two parts : one long and one short. In how many ways can  $n$  pairs be formed such that each pair consists of one short part and one long part ?
28. A candy company puts 10 chocolate bars in a bag, chosen from three different types.
  - (a) In how many ways can this set be formed ?
  - (b) How many sets contain at least one bar of each type ?
29. Let  $A$  be a set with  $n$  elements.
  - (a) Find the number of relations that can be defined on  $A$ .

- (b) Find the number of relations  $R$  on  $A$  that are : (a) Reflexive, (b) Symmetric, (c) Reflexive and Symmetric, (d) Antisymmetric.
30. Find the number of vectors  $v = (a_1, a_2, \dots, a_n)$  in the following cases :
- (a)  $a_i \in \{0, 1, \dots, k-1\}$  for all  $i = 1, 2, \dots, n$ .
- (b)  $a_i \in \{0, 1, \dots, k_i-1\}$  for all  $i = 1, 2, \dots, n$ .
- (c)  $a_i \in \{0, 1\}$  for all  $i = 1, 2, \dots, n$  and  $\sum_{i=1}^n a_i = r$ .
31. If  $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$  is the prime factorization of  $n$ , find :
- (a) The number of divisors of  $n$ .
- (b) The number of divisors of  $n$  that are not divisible by any perfect square other than 1.
32. If  $p$  is a prime number, prove that  $p$  divides  $\binom{p}{k}$  for any integer  $k$  such that  $0 < k < p$ .

## Binomial Identities

### 1.4 The Binomial Theorem

In this section, we present the Binomial Theorem as an application of combinations, along with the Multinomial Theorem which generalizes it.

#### Theorem (1.4) : Kharji-Pascal's Identity

For any integers  $n \geq k \geq 1$ , the following identity holds :

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (1)$$

*Démonstration.* Let  $A = \{a_1, a_2, \dots, a_n\}$  be a set of size  $n$ , and let  $B$  be a subset of  $A$  of size  $k$ . There are two cases :

1.  $a_n \notin B$  : The number of such subsets is equal to the number of ways to choose  $k$  elements from the set  $A \setminus \{a_n\}$ , which is  $\binom{n-1}{k}$ .
2.  $a_n \in B$  : The number of such subsets is equal to the number of ways to choose  $k-1$  elements from the set  $A \setminus \{a_n\}$ , which is  $\binom{n-1}{k-1}$ .

According to the **Rule of Sum**, the total number of subsets of size  $k$  is the sum of these two cases :

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

□

### Pascal's Identity and the Binomial Theorem

Using Pascal's Identity, we can construct **Pascal's Triangle**, which consists of the values of  $\binom{n}{k}$  :



where the **Generalized Binomial Coefficients** are given by :

$$\binom{k}{n} = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \frac{k(k-1)\dots(k-n+1)}{n!} & n > 0 \end{cases}$$

For a positive integer  $m$ , the expansion of  $(1-x)^{-m}$  is :

$$(1-x)^{-m} = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n$$

**Proof of Generalized Binomial Expansion :**

For a positive integer  $m$ , the expansion of

$$(1-x)^{-m}$$

is derived as follows :

$$\begin{aligned} (1-x)^{-m} &= \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \binom{-m}{n} x^n \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(-m)(-m-1)\dots(-m-n+1)}{n!} x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{m(m+1)\dots(m+n-1)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n \end{aligned}$$

**Example (1.8)**

Find the expansion of  $(x+y)^3$ .

**Solution :** Using the binomial coefficients  $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$  :

$$(x+y)^3 = \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3 = x^3 + 3x^2 y + 3x y^2 + y^3$$

**Example (1.9)**

Prove that  $2^n = \sum_{k=0}^n \binom{n}{k}$ .

**Solution :** From the Binomial Theorem, let  $x = 1$  and  $y = 1$  :

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$$

### Example (1.10)

Prove that :

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}$$

**Solution :** We know from the Binomial Theorem that :

$$0 = (1 - 1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \dots$$

Rearranging the terms, we get :

$$\binom{n}{0} + \binom{n}{2} + \dots = \binom{n}{1} + \binom{n}{3} + \dots$$

From Example (1.9), the total sum of all coefficients is  $2^n$ . Therefore, each side of the equation above must be exactly half of the total sum :

$$\frac{2^n}{2} = 2^{n-1}$$

## Permutations of Multisets

Every sequence formed by using all elements of a multiset  $A$  is called a **permutation** of  $A$ .

**Example :** The permutations of the multiset  $A = \{a, b, b\}$  are :

*abb, bab, and bba.*

### Theorem (1.6)

If we have  $n$  elements of  $k$  different types, such that  $r_i$  is the number of elements of type  $i$  (for  $1 \leq i \leq k$ ), then the number of distinct permutations of these elements is :

$$\frac{n!}{r_1!r_2!\dots r_k!} \quad (2)$$

**Proof :** We have  $n$  positions to fill.

1. We choose  $r_1$  positions for the elements of the first type in  $\binom{n}{r_1}$  ways.
2. We choose  $r_2$  positions for the elements of the second type from the remaining  $n - r_1$  positions in  $\binom{n - r_1}{r_2}$  ways.
3. We continue this process for all  $k$  types.

According to the **Rule of Product**, the total number of permutations is :

$$\begin{aligned} \binom{n}{r_1} \binom{n - r_1}{r_2} \dots \binom{n - r_1 - \dots - r_{k-1}}{r_k} &= \frac{n!}{r_1!(n - r_1)!} \cdot \frac{(n - r_1)!}{r_2!(n - r_1 - r_2)!} \dots \frac{r_k!}{r_k!0!} \\ &= \frac{n!}{r_1!r_2!\dots r_k!} \end{aligned}$$

■

### Example (1.11)

How many distinct permutations can be formed from the letters of the word ABACDDEFA?

**Solution :** The word consists of a total of  $n = 9$  letters. The frequency of each letter is as follows :

- The letter **A** appears 3 times ( $r_1 = 3$ ).
- The letter **B** appears 1 time ( $r_2 = 1$ ).
- The letter **C** appears 1 time ( $r_3 = 1$ ).
- The letter **D** appears 2 times ( $r_4 = 2$ ).
- The letter **E** appears 1 time ( $r_5 = 1$ ).
- The letter **F** appears 1 time ( $r_6 = 1$ ).

According to Theorem (1.6), the number of distinct permutations is :

$$\frac{9!}{3! \cdot 1! \cdot 1! \cdot 2! \cdot 1! \cdot 1!} = \frac{362,880}{6 \cdot 2} = 30,240$$

### Theorem (1.7) (The Multinomial Theorem)

If  $x_1, x_2, \dots, x_m$  are real numbers and  $n$  is a non-negative integer, then :

$$(x_1 + x_2 + \dots + x_m)^n = \sum_{r_1+r_2+\dots+r_m=n} \binom{n}{r_1, r_2, \dots, r_m} x_1^{r_1} x_2^{r_2} \dots x_m^{r_m}$$

where the multinomial coefficient is defined as :

$$\binom{n}{r_1, r_2, \dots, r_m} = \frac{n!}{r_1! r_2! \dots r_m!}$$

**Proof :**

$$(x_1 + x_2 + \dots + x_m)^n = \underbrace{(x_1 + x_2 + \dots + x_m) \cdots (x_1 + x_2 + \dots + x_m)}_{n \text{ times}}$$

Any term in the expansion is of the form  $x_1^{r_1} x_2^{r_2} \dots x_m^{r_m}$  where  $r_1, r_2, \dots, r_m$  are non-negative integers satisfying  $r_1 + r_2 + \dots + r_m = n$ . The coefficient of this term is the number of permutations of  $n$  elements, with  $r_1$  elements of type  $x_1$ ,  $r_2$  of type  $x_2$ , and  $r_m$  of type  $x_m$ . From Theorem (1.6), the coefficient of  $x_1^{r_1} x_2^{r_2} \dots x_m^{r_m}$  is :

$$\binom{n}{r_1, r_2, \dots, r_m} \blacksquare$$

**Note that :** if we set  $m = 2$  in Theorem (1.7), we obtain the **Binomial Theorem**. Furthermore, the number of distinct terms in the expansion of  $(x_1 + x_2 + \dots + x_m)^n$  is  $\binom{m-1+n}{n}$ . This is because it corresponds to the number of non-negative integer solutions to the equation  $r_1 + r_2 + \dots + r_m = n$ , as will be proven later in Corollary (1.10).

### Example (1.11)

Find the expansion of  $(x + y + z)^2$ .

**Solution :** The number of terms in the expansion is  $\binom{3-1+2}{2} = \binom{4}{2} = 6$ . From the Multinomial Theorem :

$$\begin{aligned}(x+y+z)^2 &= \binom{2}{2,0,0}x^2 + \binom{2}{0,2,0}y^2 + \binom{2}{0,0,2}z^2 + \\ &\quad \binom{2}{1,1,0}xy + \binom{2}{1,0,1}xz + \binom{2}{0,1,1}yz \\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz\end{aligned}$$

## Exercises (1.2)

- Use the Binomial Theorem to expand  $(x-2)^5$ .
- Use Pascal's Triangle to expand  $(x+1)^6$ .
- Use the Multinomial Theorem to expand  $(x+y+z)^4$ .
- What is the coefficient of  $x^3y^2z^5$  in the expansion of  $(x+y+z)^{10}$ ?
- How many terms are in the expansion of  $(x+y+z)^{70}$ ?
- Using the Binomial Theorem, prove that :  $\sum_{k=0}^n \binom{n}{k} 2^k = 3^n$ .
- Find the value of  $x$  if  $\sum_{k=0}^{50} \binom{50}{k} 8^k = x^{100}$ .
- How many permutations of the letters in the word MISSISSIPPI are there such that no two I's are adjacent?
- How many permutations of the letters in the word ILLINOIS are there such that I does not appear to the left of L?
- How many binary sequences of length  $n$  contain an even number of zeros and an odd number of ones?
- How many binary sequences of length  $n$  contain an even number of zeros and an even number of ones?
- For any positive integer  $n$ , prove that :  $\sum_{k=1}^n k \binom{n}{k} 2^{n-k} = n3^{n-1}$ .
- Give a combinatorial proof for :  $\binom{2n}{2} = 2\binom{n}{2} + n^2$ .
- Give a combinatorial proof for :  $\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$ .
- Give a combinatorial proof for :  $\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \dots + \binom{n}{r}$ .
- Prove that :  $\binom{r}{k} - \binom{r}{k+1} + \binom{r}{k+2} - \dots + (-1)^{r-k} \binom{r}{r} = \binom{r-1}{k-1}$ .  
(Hint : Use Pascal's Identity).
- Expand  $(1+1)^p = 2^p$ , then prove that  $p$  divides  $(2^p-2)$ , where  $p$  is a prime number.
- Prove the following identity (Vandermonde's Convolution Formula) :

$$\binom{m+n}{k} = \binom{m}{0} \binom{n}{k} + \binom{m}{1} \binom{n}{k-1} + \dots + \binom{m}{k} \binom{n}{0}$$

## 1.5 The Distribution Model of Counting

This section aims to present the **Distribution Model of Counting**. Suppose we want to distribute  $r$  balls into  $n$  boxes. In this context, three important questions arise :

1. Are the balls distinct or identical?
2. Is it possible for a box to contain more than one ball?
3. Are the boxes distinct or identical?

In this section, we assume that the boxes are **distinct**. This leads to four specific cases, which are illustrated in the following example.

### Example (1.12)

The table below illustrates all possible distributions of two balls into three distinct boxes.

	At most one ball per box	No conditions on the number of balls
<b>Distinct Balls</b> $(b_1, b_2)$	$[b_1][b_2][ ]$ , $[b_1][ ] [b_2]$ $[b_2][b_1][ ]$ , $[ ] [b_1][b_2]$ $[b_2][ ] [b_1]$ , $[ ] [b_2][b_1]$	$[b_1, b_2][ ][ ]$ , $[ ] [b_1, b_2][ ]$ $[ ][ ] [b_1, b_2]$ <i>+ the 6 cases on the left</i>
<b>Identical Balls</b> $(b, b)$	$[b][b][ ]$ $[b][ ] [b]$ $[ ] [b][b]$	$[b, b][ ][ ]$ $[ ] [b, b][ ]$ $[ ][ ] [b, b]$ <i>+ the 3 cases on the left</i>

### Distribution of Distinct Balls

Suppose we distribute  $r$  distinct balls  $\{b_1, b_2, \dots, b_r\}$  into  $n$  distinct boxes  $\{a_1, a_2, \dots, a_n\}$ . We can form a sequence  $x_1, x_2, \dots, x_r$  of length  $r$  from the set of boxes, where  $x_i$  is the box containing ball  $b_i$  for  $1 \leq i \leq r$ .

Conversely, given a sequence  $x_1, x_2, \dots, x_r$  of length  $r$  from the set of boxes, we can define a distribution by placing ball  $b_i$  into box  $x_i$ . Therefore, distributing  $r$  distinct balls into  $n$  distinct boxes corresponds to sequences of length  $r$  from a set of size  $n$ .

Similarly, distributing  $r$  distinct balls into  $n$  distinct boxes such that no box contains more than one ball corresponds to  $r$ -permutations from the set of boxes. From Theorem (1.1), we obtain :

### Theorem (1.8)

- (a) The number of ways to distribute  $r$  distinct balls into  $n$  distinct boxes is  $n^r$ .
- (b) The number of ways to distribute  $r$  distinct balls into  $n$  distinct boxes such that no box contains more than one ball is  $(n)_r$ .

## Distribution of Identical Balls

Suppose we have  $r$  identical balls distributed into  $n$  distinct boxes  $\{a_1, a_2, \dots, a_n\}$ . For each distribution, we take an unordered sample  $\{x_1, x_2, \dots, x_r\}$  from the set of boxes, where  $x_i$  are the boxes containing the balls.

Conversely, every unordered sample of size  $r$  corresponds to a distribution of  $r$  identical balls into  $n$  distinct boxes : the number of balls in box  $a_i$  equals the number of times  $a_i$  appears in the sample. Thus, there is a bijection between these distributions and unordered samples of length  $r$  from a set of size  $n$ .

Note that :

- If each box contains at most one ball, the samples are **sets** (combinations).
- If there are no restrictions on the number of balls per box, the samples are **multisets** (combinations with repetition).

From Theorems (1.2) and (1.3), we conclude the results for identical balls.

### Theorem (1.9)

- (a) The number of ways to distribute  $r$  identical balls into  $n$  distinct boxes is :

$$\binom{n-1+r}{r}$$

- (b) The number of ways to distribute  $r$  identical balls into  $n$  distinct boxes such that no box contains more than one ball is :

$$\binom{n}{r}$$

### Corollary (1.10)

For any integer  $r \geq 0$ , the number of non-negative integer solutions to the equation  $X_1 + X_2 + \dots + X_n = r$  is equal to :

$$\binom{n-1+r}{r}$$

**Proof :** We can view the variables  $X_1, X_2, \dots, X_n$  as  $n$  distinct boxes. Any non-negative integer solution to the equation can be seen as a distribution of  $r$  identical balls into these boxes. Conversely, from Theorem (1.9), the number of non-negative integer solutions is

$$\binom{n-1+r}{r}. \quad \blacksquare$$

### Example (1.13)

How many non-negative integer solutions does the equation  $X_1 + X_2 + X_3 = 2$  have?

**Solution :** From Corollary (1.10), the number of solutions is  $\binom{3-1+2}{2} = \binom{4}{2} = 6$ .

### Example (1.14)

Find the number of integer solutions for  $X_1 + X_2 + X_3 = 30$  given that  $X_1 \geq 3$ ,  $X_2 \geq 5$ , and  $X_3 > 6$ .

**Solution :** Since the solutions are integers,  $X_3 > 6$  is equivalent to  $X_3 \geq 7$ . Let  $Y_1 = X_1 - 3$ ,  $Y_2 = X_2 - 5$ , and  $Y_3 = X_3 - 7$ . Note that :

$$Y_1 + Y_2 + Y_3 = X_1 - 3 + X_2 - 5 + X_3 - 7 = 30 - 15 = 15$$

The number of integer solutions to the original equation is equal to the number of non-negative integer solutions to  $Y_1 + Y_2 + Y_3 = 15$ . According to Corollary (1.10), the number of solutions is :

$$\binom{3-1+15}{15} = \binom{17}{15} = \binom{17}{2} = \frac{17 \times 16}{2} = 136$$

## 1.6 Set Partitions

In this section, we find the number of partitions of a finite set of  $n$  elements into  $k$  parts, known as **Stirling numbers of the second kind**. We will show that this is equivalent to distributing  $n$  distinct balls into  $k$  identical boxes such that each box contains at least one ball.

Let  $A$  be a non-empty set. we say that  $\{A_1, A_2, \dots, A_k\}$  is a **partition** of  $A$  into  $k$  parts if :

1.  $A_i \neq \emptyset$  and  $A_i \subseteq A$  for all  $1 \leq i \leq k$ .
2.  $A_1 \cup A_2 \cup \dots \cup A_k = A$ .
3.  $A_i \cap A_j = \emptyset$  for  $1 \leq i \neq j \leq k$ .

Let  $X = \{b_1, b_2, b_3\}$  be a set of three distinct balls. The partition  $\{\{b_1, b_2\}, \{b_3\}\}$  can be viewed as a distribution of the balls  $b_1, b_2, b_3$  into two identical boxes, where  $b_1, b_2$  are in one box and  $b_3$  is in the other.

In general, if  $X$  is a finite set of  $n$  elements, every partition into  $k$  parts corresponds to a distribution of  $n$  distinct balls into  $k$  identical boxes such that each box contains at least one ball. Conversely, any distribution of  $n$  distinct balls into  $k$  identical boxes where each box contains at least one ball corresponds to a partition of the set  $X$  into  $k$  parts. ( i. e. : Any distribution of  $n$  distinct balls into  $k$  identical boxes such that each box contains at least one ball corresponds to a partition of the set  $X$  into  $k$  parts).

The number of partitions of a set of size  $n$  into  $k$  parts is denoted by  $S(n, k)$ , which are called **Stirling numbers of the second kind**.

Let  $X = \{x_1, x_2, \dots, x_n\}$ . Note that  $\{X\}$  is the only partition of  $X$  into 1 part; hence  $S(n, 1) = 1$ . Similarly,  $\{\{x_1\}, \{x_2\}, \dots, \{x_n\}\}$  is the only partition into  $n$  parts; hence  $S(n, n) = 1$ .

### Example (1.15)

Find  $S(4, 2)$ .

**Solution :** Let  $X = \{a, b, c, d\}$ . The partitions of  $X$  into 2 parts are :

$$\begin{aligned} & \{\{a\}, \{b, c, d\}\}, \quad \{\{b\}, \{a, c, d\}\}, \quad \{\{c\}, \{a, b, d\}\}, \quad \{\{d\}, \{a, b, c\}\} \\ & \{\{a, b\}, \{c, d\}\}, \quad \{\{a, c\}, \{b, d\}\}, \quad \{\{a, d\}, \{b, c\}\} \end{aligned}$$

Therefore,  $S(4, 2) = 7$ .

### Example (1.16)

For any positive integer  $n$ , show that  $S(n, n - 1) = \binom{n}{2}$ .

**Proof :** Let  $P$  be a partition of a set  $X$  of size  $n$  into  $n - 1$  parts. Note that exactly one part must consist of two elements, while each of the other parts consists of exactly one element. Thus, every partition is uniquely determined by identifying the part containing two elements. The number of partitions of  $X$  into  $n - 1$  parts is therefore equal to the number of subsets of  $X$  of size 2, which is  $\binom{n}{2}$ .

### Example (1.17)

Find  $S(4, 3)$ .

**Solution :** From Example (1.16) above :

$$S(4, 3) = \binom{4}{2} = 6$$

Since the parts in a partition must be pairwise disjoint and non-empty,  $S(n, k) = 0$  for any positive integers  $n < k$ . We define  $S(0, 0) = 1$  and  $S(n, 0) = 0$  for any positive integer  $n$ . These values are used to calculate Stirling numbers of the second kind using the following theorem.

### Theorem (1.12)

For any two positive integers  $n, k$ , then :

$$S(n + 1, k) = S(n, k - 1) + kS(n, k) \tag{3}$$

**Proof :** Let  $N = \{1, 2, \dots, n\}$  and  $N' = \{1, 2, \dots, n, n + 1\}$ . Any partition of the set  $N'$  into  $k$  parts can be obtained in a unique way from the following :

1. A partition of the set  $N$  into  $k - 1$  parts, by adding the set  $\{n + 1\}$  as a separate part to that partition. The number of partitions in this case is  $S(n, k - 1)$ .

2. A partition of the set  $N$  into  $k$  parts, by selecting one of the  $k$  parts and then adding the element  $n + 1$  to it. According to the product principle, the number of partitions in this case is  $kS(n, k)$ .

From the sum principle, it follows that :

$$S(n + 1, k) = S(n, k - 1) + kS(n, k) \quad \blacksquare$$

## Example (1.18)

Using Theorem (1.12), find  $S(5, 3)$ .

**Solution :**

$$S(5, 3) = S(4, 2) + 3S(4, 3) = 7 + 3 \cdot 6 = 25$$

The following table illustrates a method to find Stirling numbers of the second kind using Theorem (1.12).

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$n = 1$	1	0	0	0	0	0	0
$n = 2$	1	1	0	0	0	0	0
$n = 3$	1	3	1	0	0	0	0
$n = 4$	1	7	6	1	0	0	0
$n = 5$	1	15	25	10	1	0	0
$n = 6$	1	31	90	65	15	1	0
$n = 7$	1	63	301	350	140	21	1

## Theorem (1.13)

The number of surjective functions from a set with  $m$  elements to a set with  $n$  elements, where  $m \geq n$ , is equal to  $n!S(m, n)$ .

**Proof :** Let  $X = \{x_1, x_2, \dots, x_m\}$  and  $Y = \{y_1, y_2, \dots, y_n\}$ , and let  $f : X \rightarrow Y$  be a surjective function. For each  $1 \leq i \leq n$ , we define the set  $f^{-1}(y_i) = \{x \in X : f(x) = y_i\}$ . It is clear that  $f^{-1}(y_i) \subseteq X$  and that  $f^{-1}(y_i) \cap f^{-1}(y_j) = \emptyset$  if  $i \neq j$ . Also,  $f^{-1}(y_i) \neq \emptyset$  because the function is surjective. Thus,  $\{f^{-1}(y_i) : 1 \leq i \leq n\}$  is a partition of set  $X$  into  $n$  parts. Conversely, we can obtain  $n!$  surjective functions for any partition of  $X$  into  $n$  parts.

of the set  $X$ . Since the number of partitions of set  $X$  into  $n$  parts is  $S(m, n)$ , it follows from the product principle that the number of surjective functions is  $n!S(m, n)$ .  $\blacksquare$

Stirling numbers can also be obtained from the following theorem, which we will prove using Theorem (1.13) after calculating the number of surjective functions in another way in Theorem (2.2) in Chapter Two.

## Theorem (1.14)

For any two positive integers  $n, k$  such that  $n \geq k$  :

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n \quad (4)$$

We denote the total number of partitions of a set of size  $n$  by the symbol  $B_n$ . It is clear that :

$$B_n = \sum_{k=0}^n S(n, k) \quad (5)$$

These numbers are called **Bell numbers**. For example,  $B_4 = 15$ .

$B_n$  can be obtained from the table of Stirling numbers of the second kind by summing the elements of row  $n$ .

## 1.7 Partitions of Integers

Let  $n$  be a positive integer. We say that a non-increasing sequence of positive integers  $n_1, n_2, \dots, n_k$  is a **partition** of  $n$  if  $n = n_1 + n_2 + \dots + n_k$ , and for each  $1 \leq i \leq k$ , we call  $n_i$  a **part**. For example, 3, 3, 2, 1 is a partition of the number 9.

We denote the number of partitions of  $n$  by  $p(n)$ . We also denote the number of partitions of  $n$  into  $k$  parts by  $p_k(n)$ .

### Example (1.19)

The partitions of the number 5 are :

1, 1, 1, 1, 1  
2, 1, 1, 1  
2, 2, 1  
3, 1, 1  
3, 2  
4, 1  
5

Accordingly,  $p(5) = 7$ , and also :

$$p_1(5) = 1, \quad p_2(5) = 2, \quad p_3(5) = 2, \quad p_4(5) = 1, \quad p_5(5) = 1$$

It is clear that :  $p(n) = \sum_{k=1}^n p_k(n)$ .

Note that it is possible to view the partition  $n_1, n_2, \dots, n_k$  of the positive integer  $n$  as a distribution of  $n$  identical balls into  $k$  identical boxes such that the boxes contain  $n_1, n_2, \dots, n_k$  balls. Likewise, the distribution of  $n$  identical balls into  $k$  identical boxes

such that no box is empty can be viewed as a partition of  $n$  into  $k$  parts. Therefore, there is a bijection between the partitions of  $n$  and the distributions of  $n$  identical balls into identical boxes such that no box is empty.

It is easy to verify that :

$$p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, p(6) = 11, p(7) = 15$$

It can also be observed that :

$$p_n(n) = p_1(n) = p_{n-1}(n) = 1$$

## Theorem (1.15)

For any two positive integers  $n, k$  :

$$p_k(n+k) = p_1(n) + p_2(n) + \cdots + p_k(n) = \sum_{i=1}^k p_i(n) \quad (6)$$

**Proof :** The right-hand side is the number of partitions of  $n$  whose number of parts is less than or equal to  $k$ . Let  $n_1, n_2, \dots, n_i$  be a partition of  $n$  into  $i$  parts where  $i \leq k$ . From this partition, we construct a partition of the number  $n+k$  into  $k$  parts as follows : we add 1 to each  $n_j$ , and then we add a sequence where each term is 1 with a length of  $k-i$ , thus obtaining :

$$n_1 + 1, n_2 + 1, \dots, n_i + 1, 1, 1, \dots, 1$$

where :

$$(n_1 + 1) + (n_2 + 1) + \cdots + (n_i + 1) + (k - i) = n_1 + n_2 + \cdots + n_i + k = n + k$$

Conversely, from any partition of the number  $n+k$  into  $k$  parts, we can form a partition of the number  $n$  where the number of parts is less than or equal to  $k$  by removing all parts equal to 1 and then subtracting 1 from the other parts. Consequently,  $p_k(n+k) = \sum_{i=1}^k p_i(n)$ . ■

## Corollary (1.16)

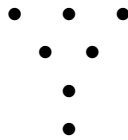
$$p_k(m) = \sum_{i=1}^k p_i(m-k) \quad (7)$$

**Proof :** Set  $n = m - k$  in Theorem (1.15). ■

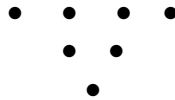
The following table provides the values of  $p_k(n)$  when  $1 \leq k, n \leq 10$ . It has been constructed based on Corollary (1.16) and the fact that  $p_k(n) = 0$  when  $k > n$ .

For the partition  $n_1, n_2, \dots, n_k$  of a positive integer  $n$ , we construct a **Ferrers diagram** by drawing  $n_i$  dots in row  $i$  for each  $1 \leq i \leq k$ . For example, the Ferrers diagram for the partition 3, 2, 1, 1 of the number 7 is shown below :

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
$n = 1$	1	0	0	0	0	0	0	0	0	0
$n = 2$	1	1	0	0	0	0	0	0	0	0
$n = 3$	1	1	1	0	0	0	0	0	0	0
$n = 4$	1	2	1	1	0	0	0	0	0	0
$n = 5$	1	2	2	1	1	0	0	0	0	0
$n = 6$	1	3	3	2	1	1	0	0	0	0
$n = 7$	1	3	4	3	2	1	1	0	0	0
$n = 8$	1	4	5	5	3	2	1	1	0	0
$n = 9$	1	4	7	6	5	3	2	1	1	0
$n = 10$	1	5	8	9	7	5	3	2	1	1



Similarly, the Ferrers diagram for the partition 4, 2, 1 of the number 7 is :



For each Ferrers diagram of a partition of the number  $n$ , we can obtain its **transpose** by converting rows into columns. Note that what we will obtain is also a Ferrers diagram of a partition of the same number  $n$ .

### Example (1.20)

The partitions of the number 4 into parts, each of which is less than or equal to 3, are :

3, 1  
 2, 2  
 2, 1, 1  
 1, 1, 1, 1

The corresponding partitions obtained from the transpose of the Ferrers diagrams shown above are, respectively :

2, 1, 1  
 2, 2  
 3, 1  
 4

We notice that the number of partitions of the number 4 into parts, each of which is at most 3, is equal to the number of partitions of the number 4 into three parts or fewer. In fact, this can be generalized as in the following theorem :

## Theorem (1.17)

The number of partitions of a positive integer  $n$  into parts, each of which is at most  $k$ , is equal to the number of partitions of  $n$  into at most  $k$  parts.

**Proof :** Note that the transpose of the Ferrers diagram of a partition of  $n$  into parts each less than or equal to  $k$  provides the Ferrers diagram of a partition of  $n$  into at most  $k$  parts. This is true because the maximum number of dots in a row in the first diagram is at most  $k$ , and therefore the number of rows in the transpose is at most  $k$ . Conversely, the transpose of the Ferrers diagram of a partition of  $n$  into at most  $k$  parts is the Ferrers diagram of a partition of  $n$  into parts each less than or equal to  $k$ .

## Exercises (1.3)

- Write an equivalent statement (in the form of the number of integer solutions to an equation) for the following :
  - The number of ways to distribute  $r$  identical balls into  $n$  distinct boxes.
  - The number of ways to distribute  $r$  identical balls into  $n$  distinct boxes such that each box contains at most two balls.
  - The number of subsets consisting of three elements from the set  $\{A, B, C, D, E\}$ .
  - Find the number of ways to distribute  $r$  identical balls into  $n$  distinct boxes such that each box contains at least two balls.
- Find the number of integer solutions to the equation  $X_1 + X_2 + X_3 + X_4 + X_5 = 15$  if  $X_k \geq 0$  for all  $1 \leq k \leq 5$ .
- Find the number of integer solutions to the equation  $X_1 + X_2 + X_3 + X_4 + X_5 = 15$  if  $X_k \geq -3$ ?
- Find the number of integer solutions to the equation  $X_1 + X_2 + X_3 + X_4 + X_5 = 15$  if  $X_k \geq 2k$ ?
- Find the number of integer solutions to the inequality  $X_1 + X_2 + X_3 \leq 10$  if  $X_k \geq 0$ ?
- Find the number of integer solutions to the inequality  $X_1 + X_2 + X_3 \leq 10$  if  $X_k \geq -2$ ?
- Find the number of integer solutions to the inequality  $X_1 + X_2 + X_3 \leq 10$  if  $X_k \geq 2k$ ?
- Find the number of integer solutions for the system of equations  $X_1 + X_2 + X_3 + X_4 + X_5 = 20$  and  $X_1 + X_2 + X_3 = 5$ , where  $X_i \geq 0$ .
- For any positive integer  $n \geq 2$ , prove that :  $S(n, 2) = 2^{n-1} - 1$ .
- For any positive integer  $n \geq 3$ , prove that :  $S(n, 3) = \frac{1}{2}(3^{n-1} + 1) - 2^{n-1}$ .
- For any positive integer  $n \geq 4$ , prove that :

$$S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$$

12. For any positive integer  $n \geq 6$ , prove that :

$$S(n, n-3) = \binom{n}{4} + \binom{n}{3} \binom{n-3}{2} + \frac{1}{6} \binom{n}{2} \binom{n-2}{2} \binom{n-4}{2}$$

13. Let  $X = \{a, b, c, d, e\}$  and  $Y = \{f, g, h, i\}$ . How many surjective (onto) functions are there from  $X$  to  $Y$ ?

14. Write down all partitions for the following numbers : 3, 4, 6, 7.

15. For any positive integer  $n$ , prove that :  $P_2(n) = \lfloor \frac{n}{2} \rfloor$ .

16. For any positive integer  $n$ , prove that :  $P_n(2n) = P(n)$ .

17. For any positive integer  $n$ , prove that :  $P_n(2n+1) = P(n+1) - 1$ .

18. For any positive integer  $n \geq 4$ , prove that :  $P_{n-2}(n) = 2$ .