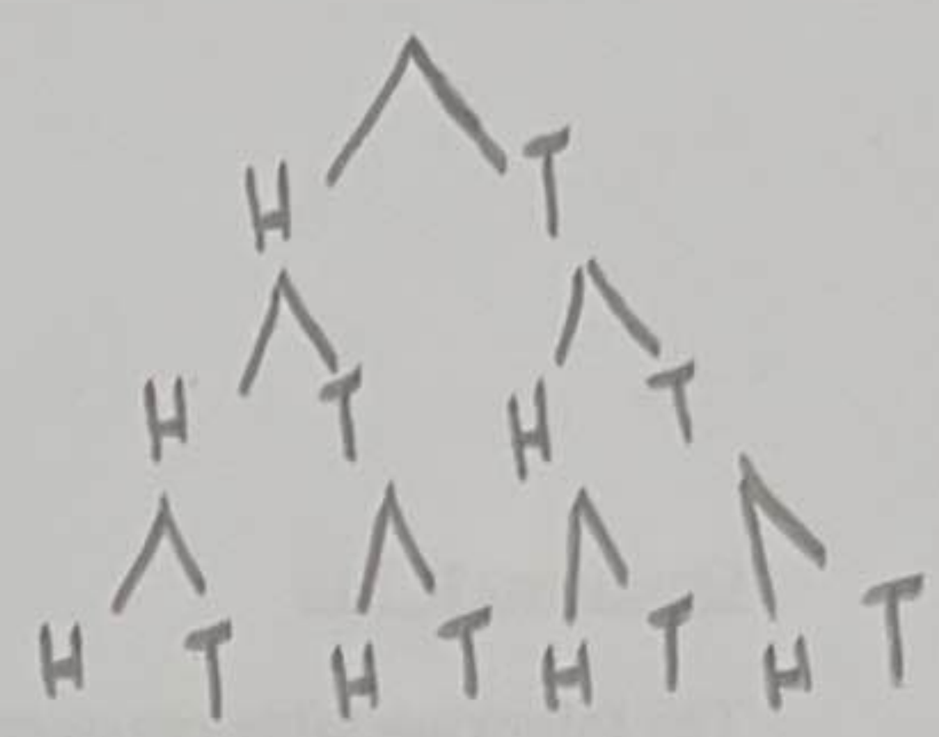
Basic probability theory - Chapter 1 :-

Ex 1 ① $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$



$n(\Omega) = 8$ $\quad \boxed{(\text{or } n(\Omega) = 2^3 = 8)}$

A = at least 2H $= \{HHH, HHT, HTH, THH\}$ $\quad n(A) = 4$
≥ 2

B = at most 2H $= \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$
≤ 2
$n(B) = 7$

C = H on second toss $= \{HHH, HHT, THH, THT\}$ $\quad n(C) = 4$

② • $P(A) = \frac{n(A)}{n(\Omega)} = \frac{4}{8} = \frac{1}{2} = 0.5$

• $P(A \cap B) = \frac{n(A \cap B)}{n(\Omega)} = \frac{3}{8} = 0.375$

$A \cap B = \{HHT, HTH, THH\}$

• $P(A \cup C) = \frac{n(A \cup C)}{n(\Omega)} = \frac{5}{8} = 0.625$

$A \cup C = \{HHH, HHT, HTH, THH, THT\}$

other solution:-

$P(A \cup C) = P(A) + P(C) - P(A \cap C)$

$= \frac{4}{8} + \frac{4}{8} - \frac{3}{8} = \frac{5}{8}$

Ex 2 $E(X+4)=10$ and $E((X+4)^2)=116$

① $E(X)=?$

$E(X+4)=10$

$\Rightarrow E[X]+4=10$

$\Rightarrow \boxed{E[X]=6}$ #

② $V(X)=?$ def. $V(X)=E((X-\mu)^2)=E(X^2)-(E(X))^2$

$V(X+4)=E[(X+4)^2]-(E[X+4])^2$

$V(X)+V(4)=116-(10)^2$

$V(X)+0=116-100$

$\boxed{V(X)=16}$ #

$\boxed{V(X+a)=V(X)}$ a constant

③ $V(X+4)=?$

$V(X+4)=V(X)+V(4)$

$=16+0$

$\boxed{=16}$ #

$\boxed{Ex\ 3}$ $P(A) = 0.4$ $P(B) = 0.5$ $P(A \cap B) = 0.3$

① $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.4 + 0.5 - 0.3$

$= 0.6$

② $P(A \cap B^c) = P(A) - P(A \cap B)$

$= 0.4 - 0.3$

$= 0.1$

③ $P(A^c \cup B^c) = P(A \cap B)^c$

$\boxed{P(A^c) = 1 - P(A)}$

$= 1 - P(A \cap B)$

$= 1 - 0.3$

$= 0.7$

other Solution

$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c)$

ديمورجان

De morgan's laws: $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$

$= 0.6 + 0.5 - P(A \cup B)^c$

$1 - P(A \cup B)$

$= 1 - 0.6$

$= 0.4$

$= 0.6 + 0.5 - 0.4$

$= 0.7$

Ex 4 $P(R) = 0.75$ / $P(I) = 0.65$ / $P(T) = 0.55$ /

$P(IR) = 0.5$ / $P(RT) = 0.4$ / $P(IT) = 0.3$

$P(RIT) = 0.20$

① $P(R \cup I \cup T) = P(R) + P(I) + P(T) - P(RI) - P(RT) - P(IT) + P(RIT)$

$= 0.75 + 0.65 + 0.55 - 0.5 - 0.4 - 0.3 + 0.20$

$= 0.95$

② $P(R^c \cup T^c) = P(R \cap T)^c$

$= 1 - P(R \cap T)$

$= 1 - 0.4$

$= 0.6$

Ex 5 X = number turning up when tossing one fair die.

①

$$P(X = x) = f(x) = \begin{cases} \frac{1}{6} & x = 1, 2, \ldots, 6 \\ 0 & o.w \end{cases}$$

discrete Uniform

$P(X=x) = \frac{1}{k}$; $x = 1, \ldots K$

②

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6} = 1$

$\boxed{\text{Ex 6}}$

$$f(x) = \begin{cases} 3x^2 & ; 0 < x < 1 \\ 0 & ; \text{o.w} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$$= \int_{0}^{x} 3t^2\, dt = \frac{\cancel{3} t^3}{\cancel{3}} \Big|_0^x = x^3 \quad ; 0 < x < 1$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$\boxed{\text{Ex 7}}$ $\quad X \sim \text{Bin}(n, p) \qquad \text{mean} = np \qquad \text{Variance} = npq$

$E(X) = 10 \quad , V(X) = 5$

$\Rightarrow \quad V(X) = \underbrace{np}\, q$

$\Rightarrow \quad V(X) = E(X)\, q \qquad \boxed{E(X) = np}$

$\Rightarrow \quad 5 = 10\, q$

$\Rightarrow \quad \boxed{q = \frac{1}{2}} \quad / \quad \boxed{p = 1 - q = \frac{1}{2}}$

$E(X) = np$

$10 = n \frac{1}{2} \quad \Rightarrow \quad \boxed{n = 20}$

$\boxed{Ex\ 7}$

① $P(X=x) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0, 1, \ldots, n$

$= \binom{20}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{20-x} \quad ; \quad x = 0, 1, \ldots, 20$

② $P(X=0) = \binom{20}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{20} = 9.5 \times 10^{-7}$

$P(X=1) = 1.9 \times 10^{-5}$

$P(X=2) = 1.8 \times 10^{-4}$

③ $P(X \geq 0) = 1 - P(X < 0)$

$= 1 - 0$

$= 1$

or $\sum_{x=0}^{20} \left((20Cx)(0.5^x)(0.5^{20-x}) \right)$

$= 1$

$\boxed{Ex\ 8}$ $X \sim Bin(n=10, \ p=0.4)$ $\quad q = 1-p$

$P(X=x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, \ldots, 10$

① $P(X=4) = 0.2508$

② $P(X<3) = P(X=0) + P(X=1) + P(X=2) = 0.1673$

③ $P(X>8) = P(X=9) + P(X=10) = 0.0017$

④ $\mu = np = 10 \times 0.4 = 4$

⑤ $\sigma^2 = npq = 10 \times 0.4 \times 0.6$

$= 2.4$

Ex 9

$$f(x) = \begin{cases} k x^2 (1-x)^9 & ; \ 0 < x < 1 \\ 0 & ; \ \text{o.w} \end{cases}$$

x^2 + $(1-x)^9$ + قيمها محصورة $0<x<1$

$$f(x) < \begin{matrix} \geq 0 \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{matrix}$$

- value of k = ??

we know: $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^1 k x^2 (1-x)^9\,dx = 1$$

let $u = 1-x$ $\quad du = -dx$

$x = 1-u$

if $x = 1 \rightarrow u = 0$

$x = 0 \rightarrow u = 1$

$$\Rightarrow \int_1^0 k (1-u)^2 u^9 \cdot -du = 1$$

$$\Rightarrow \int_0^1 k (1 - 2u + u^2) u^9\,du = 1$$

$$\Rightarrow k \int_0^1 (u^9 - 2u^{10} + u^{11})\,du = 1$$

$$\Rightarrow k \left[\frac{u^{10}}{10} - 2\frac{u^{11}}{11} + \frac{u^{12}}{12} \right]_0^1 = 1$$

$$\Rightarrow k \left[\frac{1}{10} - \frac{2}{11} + \frac{1}{12} \right] = 1$$

$$\Rightarrow k \left[\frac{1}{660} \right] = 1 \Rightarrow \boxed{k = 660}$$

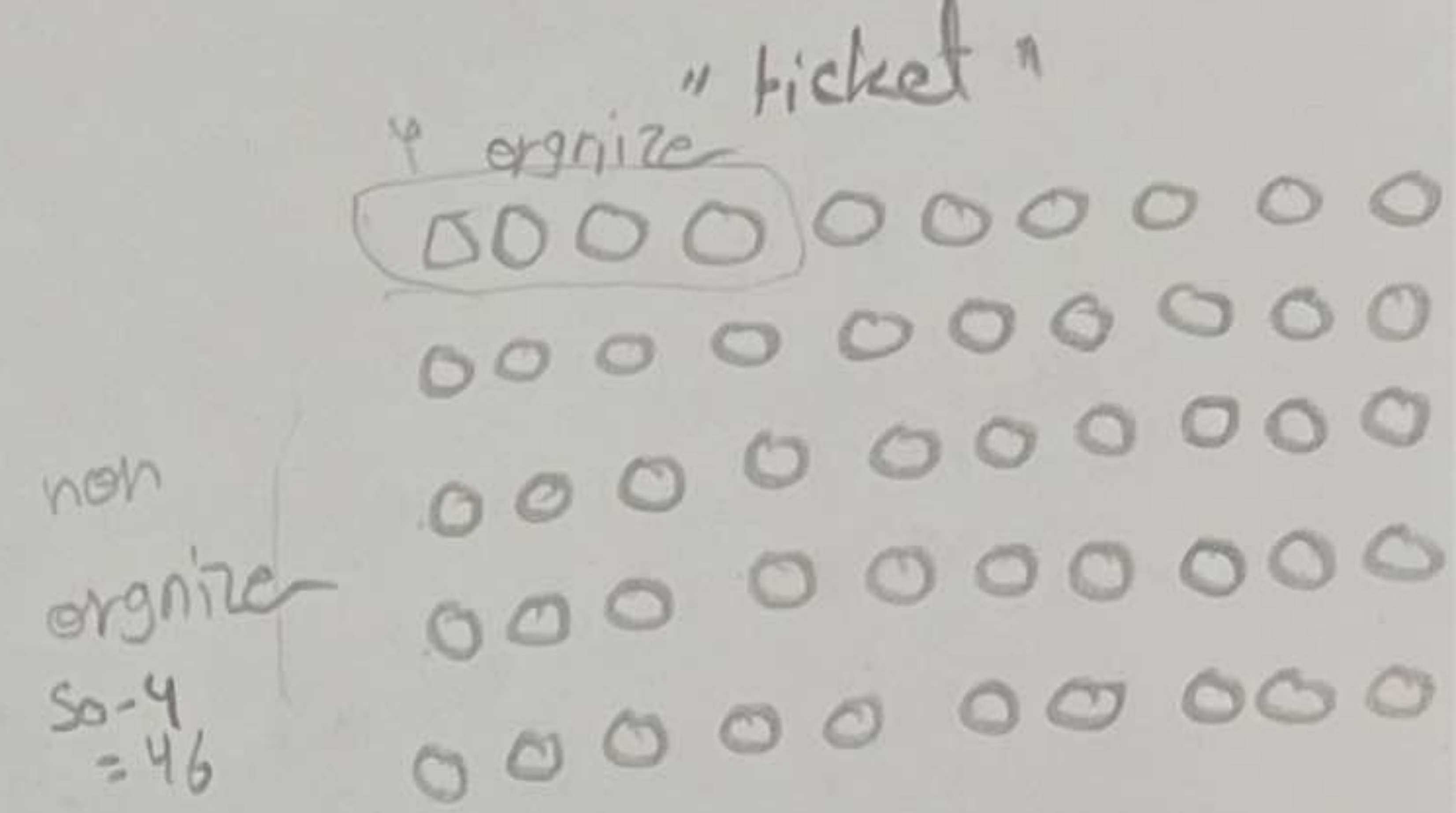
Ex 10 number of ticket = 50

number of prizes = 3

number of organizers = 4

~~number of tick~~

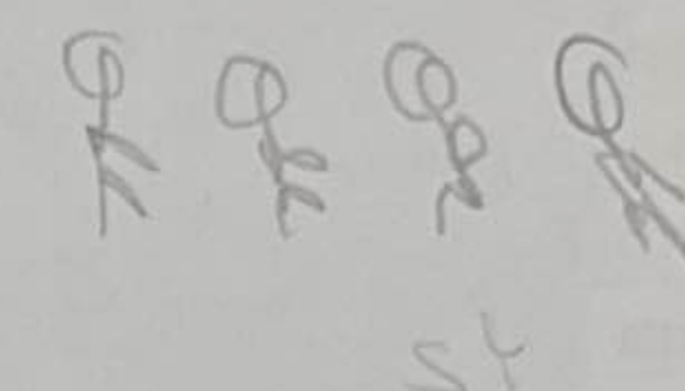
Sample space = ${}_{50}C_3 = 19600$



① all of the prizes:

$$= \frac{\binom{4}{3}\binom{46}{0}}{\binom{50}{3}} = \frac{4}{19600}$$

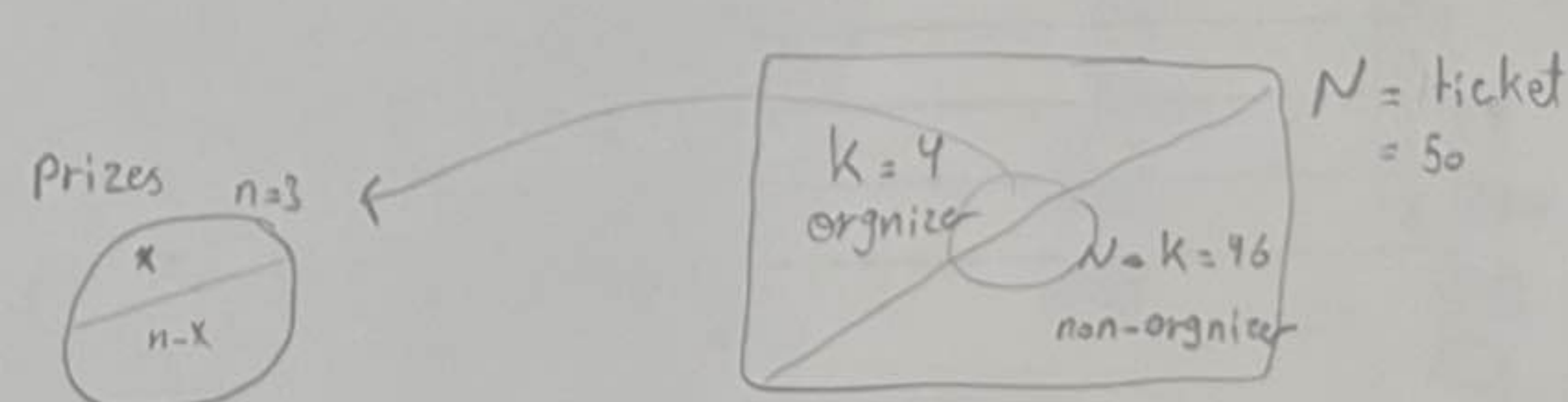
four organizes of raffle ⇒



② $\dfrac{\binom{4}{2}\binom{46}{1}}{\binom{50}{3}} = \dfrac{276}{19600}$

③ $\dfrac{\binom{4}{1}\binom{46}{2}}{\binom{50}{3}} = \dfrac{4140}{19600}$

④ $\dfrac{\binom{4}{0}\binom{46}{3}}{\binom{50}{3}} = \dfrac{1518}{19600}$



x = orginizers win

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}};$$