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Chapter 10

Multiple State Models

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10.1 Introduction

10.1.1 The Single Decrement Model

In the ordinary single decrement model, there are two states, alive (0) and dead (1):

(0) alive	$\xrightarrow{\mu_{x+t}}$	dead (1)

Model 1: The Single Decrement Model

The single decrement model is characterized by the force of mortality

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} = \frac{{}_tq_x'}{{}_tp_x} = \frac{1}{{}_tp_x}\lim_{h\to 0}\frac{{}_{t+h}q_x - {}_tq_x}{h} = \frac{1}{{}_tp_x}\lim_{h\to 0}\frac{{}_tp_x - {}_{t+h}p_x}{h} = \lim_{h\to 0}\frac{{}_hq_{x+t}}{h}.$$

in other works we can write

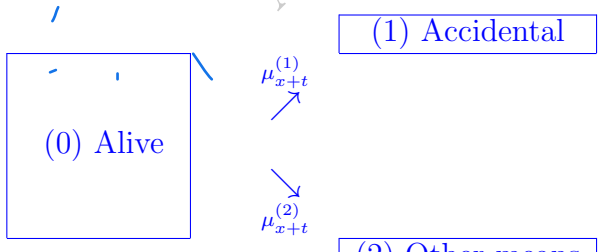
$${}_hq_x = h\mu_{x+t} + o(h) \quad \text{where} \quad \lim_{h\to 0}\frac{o(h)}{h} = 0.$$

So for $\mathrm{d}t$ small enough we use the approximation ${}_{\mathrm{d}t}q_x \simeq \mu_{x+t}\,\mathrm{d}t$.
For this model, the probability for (x) to stay in state 0 (alive) for a period of length t is

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+u}\,\mathrm{d}u\right) \quad \text{and} \quad {}_tq_x = \int_0^t {}_rp_x\,\mu_{x+r}\,\mathrm{d}r = 1 - {}_tp_x.$$

10.1.2 The Double Decrement Model

In multiple decrements, what we need are the various forces of decrement. In the double decrement model, they are 2 modes of decrements, say, accidental death (1) and death due to other causes (2).



Model 2: The double Decrement Model

In this model, we are interested in the joint distribution of:

- the time for the individual to leave state 0,
- the state that the individual would enter after leaving state 0.

The forces of decrement are the instantaneous rates of transition.
Similarly to above

$$ {}_{dt}q_x^{(1)} \simeq \mu_{x+t}^{(1)}\, dt \quad \text{and} \quad {}_{dt}q_x^{(2)} \simeq \mu_{x+t}^{(2)}\, dt. $$

Remember that $\mu_{x+t}^{(\tau)} = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)}$.
Occupancy probability (means no change in state) :

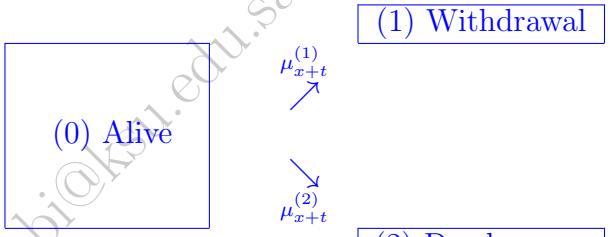
$$ {}_tp_x^{(\tau)} = \exp\left(-\int_0^t \mu_{x+u}^{(\tau)}\, du\right). $$

Transition probabilities:

$$ {}_tq_x^{(1)} = \int_0^t {}_rp_x^{(\tau)}\mu_{x+r}^{(1)}\, dr \quad \text{and} \quad {}_tq_x^{(2)} = \int_0^t {}_rp_x^{(\tau)}\mu_{x+r}^{(2)}\, dr. $$

There are two characteristics for multiple decrement models.

- Transition to any state is irreversible.
- There is no need to specify the transition intensity after decrement has happened. Even if we change the model to a withdrawal model, we would not specify what kind of decrements would the member follows after withdrawal.

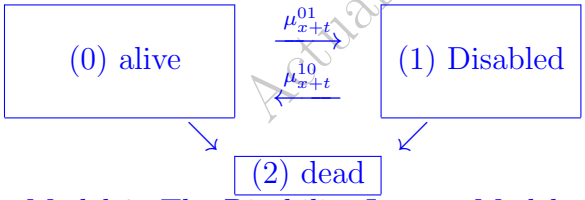


Model 2: The double Decrement Model

Of course the member would still die some time after withdrawal. But we simply do not model that because we are not interested in it.

10.1.3 The Disability Income Model

A multiple state model models the transition rates between states. Transition to any state can be reversible or irreversible. Consider the following disability income model:



Model 3: The Disability Income Model

In this model, the insured can switch between state 0 and state 1. However, transition to state 2 is irreversible. We can change the above to a permanent disability model by disallowing transition from state 1 back to state 0.

In this chapter we want to analyze models with structures similar to **Model 3**. We shall first discuss a simple version, where state changes can only occurs at the end of each period. Then we shall discuss the more general case when state changes can only occur any time.

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10.2 Discrete Markov Chain

In this section we discuss discrete–time Markov chain, a model for which transitions can only occur at the end of each period. Such a model is useful in modeling health status, bonus malus system and credit rating in group health and non–life insurances. Let the state at time t be Y_t. Let the state space, which is the set of all possible states, be E. For example, in the disability income model above, we have $E = \{0, 1, 2\}$ and $Y_0 = 0$. As time progresses, Y switches between 0 and 1 and finally get struck in state 2.

10.2.1 Transition Probability Matrix and the Markov Property

For a discrete–time Markov chain, we assume that when the process is in state i at time t, the probability that it would be in state) at time $t+1$ is $p_t^{ij} = \mathrm{P}(Y_{t+1} = j \mid Y_t = i)$. The associated transition probability matrix take the form

$$\mathbf{P}_t = \begin{pmatrix} p_t^{00} & p_t^{01} & p_t^{02} & \cdots & \cdots & \cdots & \cdots & p_t^{0n} \\ p_t^{10} & p_t^{11} & p_t^{12} & \cdots & \cdots & \cdots & \cdots & p_t^{1n} \\ p_t^{20} & p_t^{21} & p_t^{22} & \cdots & \cdots & \cdots & \cdots & p_t^{2n} \\ \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & & & \vdots \\ p_t^{n0} & p_t^{n1} & p_t^{n2} & \cdots & \cdots & \cdots & \cdots & p_t^{nn} \end{pmatrix}$$

The sum of the probabilities on each row is 1 that is $\sum_{j=0}^{n} p_t^{ij} = 1$ for all $i = 1, \ldots, n$. Moreover, we assume for any $t \geq 0$ and $i,\ j \in E$.

The Markov Property $\mathrm{P}(Y_{t+1} = j \mid Y_t = i, Y_{t-1} = i_{t-1}, Y_{t-2} = i_{t-2}, \cdots) = \mathrm{P}(Y_{t+1} = j \mid Y_t = i)$

This above says that the conditional distribution of any future state Y_{t+1}, given the past states $\{Y_0,\ Y_1, \ldots, Y_{t-1}\}$ and the present state Y_t, depends only on the present state Y_t but not the past states.

Loosely speaking, "Markov" means **given the present**, you can **forget about the past!**

- If $\mathbf{P}_t$ is constant with respect to t that is time–independent ($\mathbf{P}_{t+1} = \mathbf{P}_t = \mathbf{P}_0\ \forall\ t \geq 0$) the Markov chain is said to be **homogeneous**.
- If $\mathbf{P}_t$ is time–dependent, the Markov chain is said to be **inhomogeneous**.

In general we have the following properties

$$ {}_s\mathbf{P}_x = \mathbf{P}_x \mathbf{P}_{x+1} \cdots \mathbf{P}_{x+s-1} $$

${}_s\mathbf{P}_x$ is called the s–step transition matrix of the Markov chain and its entries are ${}_sp_x^{ij} = \mathrm{P}\left(Y_{x+s} = j \mid Y_x = i\right)$. This lead to the The Chapman–Kolmogorov (CK) Equation

$$ {}_{t+s}\mathbf{P}_x = {}_t\mathbf{P}_x\ {}_s\mathbf{P}_{x+t}. $$

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Example 10.2.1 *For a multiple state model with three states, Healthy (0), Disabled (1), and Dead (2): (i) For $k = 0, 1$, $p_{x+k}^{00} = 0.70$, $p_{x+k}^{01} = 0.20$, $p_{x+k}^{10} = 0.10$, $p_{x+k}^{12} = 0.25$. (ii) There are 100 lives at the start, all Healthy. Their future states are independent. Calculate the expectation and variance of the number of the original 100 lives who die within the first two years*

Solution: Let ${}_2D_0$ the r.v. describing the number of lives who die within the first two years. The possible values of ${}_2D_0$ are $\{0, 1, \ldots, 100\}$. The probability that one individual from the population dies within two years is ${}_2p_x^{02}$. Observe first that ${}_2D_0$ follows a binomial distribution. So $\mathrm{E}\,[{}_2D_0] = 100\ {}_2p_x^{02}$ and $\mathrm{Var}\,({}_2D_0) = 100\ {}_2p_x^{02}(1 - {}_2p_x^{02})$. To calculate ${}_2p_x^{02}$ we need to calculate the 2–step transition. We have

$$\mathbf{P}_x = \mathbf{P}_{x+1} = \begin{pmatrix} 0.70 & 0.20 & ? \\ 0.10 & ? & 0.25 \\ ? & ? & ? \end{pmatrix} = \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & 0.65 & 0.25 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}_2\mathbf{P}_x = \mathbf{P}_x\mathbf{P}_{x+1} = \mathbf{P}_x^2 = \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & 0.65 & 0.25 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0.51 & 0.27 & 0.22 \\ 0.135 & 0.4425 & 0.4225 \\ 0 & 0 & 1 \end{pmatrix}$$

therefore ${}_2p_x^{02} = 0.22$. In fact we do not need the full matrix $\mathbf{P}_x^2$, we can calculate

$${}_2p_x^{02} = \begin{pmatrix} 0.70 & 0.20 & 0.10 \end{pmatrix} \begin{pmatrix} 0.10 \\ 0.25 \\ 1 \end{pmatrix} = 0.7 \times 0.1 + 0.2 \times 0.25 + 0.1 \times 1 = 0.22.$$

Finally $E[\ {}_2D_0] = 100 \times 0.22 = 22$ and $\mathrm{Var}\,({}_2D_0) = 100 \times 0.22(1 - 0.22) = 17.16$.

10.3 Calculating APV of Cash Flows

There are mainly two kinds of cash flows associated with discrete-time Markov chain:

- Cash flow arising from being in a particular state.
- Cash flow arising from a transition from one state to another.

It is easy to calculate APV of cash flows under a discrete–time Markov chain model. Study the following example:

Example 10.3.1 *Consider a homogenous discrete–time Markov chain with transition matrix*

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix}$$

The state space is $E = \{0, 1\}$. The subject in now in state 0, and $i = 5\%$.

1. *Suppose that there is a cash flow of 1000 for being in state 0 now and the beginning of the next two years. Find the actuarial present value of the cash flow streams.*
2. *Suppose that there is a cash flow of 2000 if transition from state 1 to state 0 occurs at the end of the first, second and the third year. Find the actuarial present value of the cash flow streams.*

Solution:

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1. Let Y_t denotes the position of the homogeneous Markov chain at time t and α denotes actuarial present value of the cash flow streams:

$$\begin{aligned}\alpha &= 1000 + \frac{1000}{1.05}\mathrm{P}\left(Y_1 = 0 | Y_0 = 0\right) + \frac{1000}{(1.05)^2}\mathrm{P}\left(Y_2 = 0 | Y_0 = 0\right) \\ &= 1000\left(1 + \frac{p_0^{00}}{1.05} + \frac{{}_2p_0^{00}}{(1.05)^2}\right) = 1000\left(1 + \frac{0.4}{1.05} + \frac{0.64}{(1.05)^2}\right) = \mathbf{1961.45}\end{aligned}$$

since $p_0^{00} = p^{00} = 0.4$ and

$${}_2p_0^{00} = {}_2p^{00} = \begin{pmatrix} 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix} = 0.4 \times 0.4 + 0.6 \times 0.8 = 0.64.$$

2. Let β denotes actuarial present value of the cash flow streams:

$$\beta = 2000\left(\frac{\mathrm{P}\left(Y_0 = 1, Y_1 = 0 | Y_0 = 0\right)}{1.05} + \frac{\mathrm{P}\left(Y_1 = 1, Y_2 = 0 | Y_0 = 0\right)}{(1.05)^2} + \frac{\mathrm{P}\left(Y_2 = 1, Y_3 = 0 | Y_0 = 0\right)}{(1.05)^3}\right).$$

Calculation of the conditional probabilities: Obviously

$$\mathrm{P}\left(Y_0 = 1, Y_1 = 0 | Y_0 = 0\right) = 0,$$

and

$$\begin{aligned}\mathrm{P}\left(Y_1 = 1, Y_2 = 0 | Y_0 = 0\right) &= \frac{\mathrm{P}\left(Y_1 = 1, Y_2 = 0, Y_0 = 0\right)}{\mathrm{P}\left(Y_0 = 0\right)} \\ &= \mathrm{P}\left(Y_2 = 0 | Y_1 = 1, Y_0 = 0\right)\frac{\mathrm{P}\left(Y_1 = 1, Y_0 = 0\right)}{\mathrm{P}\left(Y_0 = 0\right)} \\ &= \mathrm{P}\left(Y_2 = 0 | Y_1 = 1\right)\mathrm{P}\left(Y_1 = 1 | Y_0 = 0\right) \\ &= p_1^{10}\, p_0^{01} = p^{10}\, p^{01} = 0.8 \times 0.6 = 0.48\end{aligned}$$

similarly

$$\begin{aligned}\mathrm{P}\left(Y_2 = 1, Y_3 = 0 | Y_0 = 0\right) &= \frac{\mathrm{P}\left(Y_3 = 0, Y_2 = 1, Y_0 = 0\right)}{\mathrm{P}\left(Y_0 = 0\right)} = \mathrm{P}\left(Y_3 = 0 | Y_2 = 1\right)\mathrm{P}\left(Y_2 = 1 | Y_0 = 0\right) \\ &= p_2^{10}\,{}_2p_0^{01} = p^{10}\,{}_2p^{01} = p^{10}\left(1 - {}_2p^{00}\right) = 0.8\left(1 - 0.64\right) = 0.288.\end{aligned}$$

Finally

$$\beta = 2000\left(\frac{0}{1.05} + \frac{0.48}{(1.05)^2} + \frac{0.288}{(1.05)^3}\right) = \mathbf{1368.3188}.$$

Example 10.3.2 *You are given the following homogenous Markov chain model with three states, namely Healthy (0), Disabled (1), and Dead (2). Transition between states occurs at the end of the year. The probabilities of moving among these various states are summarized by the transition matrix below:*

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{pmatrix}$$

An insurance company issues a 3–year insurance. A ***death benefit*** *of* $\$10,000$ *is payable at the end of the year of death. Level annual premium is payable for 3 years or until death, whichever occurs first, except that the premium is* ***waived*** *in the state of disablement. Interest rate is 6% per annum.*

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1. *Find the level premium for an insured healthy at issue* $(t = 0)$ *using E.P.*
2. *Calculate the benefit reserve at* $t = 1$ *for on the following two cases:*
   (a) *an insured healthy at* $t = 1$, *originally healthy at* $t = 0$.
   (b) *an insured disabled at* $t = 1$, *originally healthy at* $t = 0$.

Solution:

1. APV of future premiums at time 0 is

$$\mathbf{APV(FP)}_0 = P\left(1 + \frac{\mathrm{P}\,(Y_1 = 0|Y_0 = 0)}{1.06} + \frac{\mathrm{P}\,(Y_2 = 0|Y_0 = 0)}{(1.06)^2}\right) = P\left(1 + \frac{p^{00}}{1.06} + \frac{{}_2p^{00}}{(1.06)^2}\right)$$

but $p^{00} = 0.6$ and ${}_2p^{00} = \begin{pmatrix} 0.6 & 0.3 & 0.1 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0 \\ 0 \end{pmatrix} = 0.6 \times 0.6 = 0.36$. Thus

$$\mathbf{APV(FP)}_0 = P\left(1 + \frac{0.6}{1.06} + \frac{0.36}{(1.06)^2}\right) = 1.8864P.$$

APV of future benefits at time 0 is

$$\mathbf{APV(FB)}_0 = 10^4\left(\frac{\mathrm{P}\,(Y_1 = 2|Y_0 = 0)}{1.06} + \frac{\mathrm{P}\,(Y_1 \neq 2, Y_2 = 2|Y_0 = 0)}{(1.06)^2} + \frac{\mathrm{P}\,(Y_1 \neq 2, Y_2 \neq 2, Y_3 = 2|Y_0 = 0)}{(1.06)^3}\right)$$

$\mathrm{P}\,(Y_1 = 2|Y_0 = 0) = p_0^{02} = p^{02} = 0.1$, and

$$\begin{aligned} & \mathrm{P}\,(Y_1 \neq 2, Y_2 = 2|Y_0 = 0) = \mathrm{P}\,(Y_1 = 0, Y_2 = 2|Y_0 = 0) + \mathrm{P}\,(Y_1 = 1, Y_2 = 2|Y_0 = 0) \\ = \; & \mathrm{P}\,(Y_2 = 2|Y_1 = 0)\,\mathrm{P}\,(Y_1 = 0|Y_0 = 0) + \mathrm{P}\,(Y_2 = 2|Y_1 = 1)\,\mathrm{P}\,(Y_1 = 1|Y_0 = 0) \\ = \; & p^{02}\,p^{00} + p^{12}\,p^{01} = 0.1 \times 0.6 + 0.4 \times 0.3 = 0.18 \end{aligned}$$

and

$$\begin{aligned} \mathrm{P}\,(Y_1 \neq 2, Y_2 \neq 2, Y_3 = 2|Y_0 = 0) \; = \; & \mathrm{P}\,(Y_1 = 0, Y_2 = 0, Y_3 = 2|Y_0 = 0) \\ & + \mathrm{P}\,(Y_1 = 0, Y_2 = 1, Y_3 = 2|Y_0 = 0) \\ & + \mathrm{P}\,(Y_1 = 1, Y_2 = 1, Y_3 = 2|Y_0 = 0) \\ = \; & \mathrm{P}\,(Y_3 = 2|Y_2 = 0)\,\mathrm{P}\,(Y_2 = 0|Y_1 = 0)\,\mathrm{P}\,(Y_1 = 0|Y_0 = 0) \\ & + \mathrm{P}\,(Y_3 = 2|Y_2 = 1)\,\mathrm{P}\,(Y_2 = 1|Y_1 = 0)\,\mathrm{P}\,(Y_1 = 0|Y_0 = 0) \\ & + \mathrm{P}\,(Y_3 = 2|Y_2 = 1)\,\mathrm{P}\,(Y_2 = 1|Y_1 = 1)\,\mathrm{P}\,(Y_1 = 1|Y_0 = 0) \\ = \; & p^{02}\,p^{00}\,p^{00} + p^{12}\,p^{01}\,p^{00} + p^{12}\,p^{11}\,p^{01} \\ = \; & 0.1 \times 0.6 \times 0.6 \; + \; 0.4 \times 0.3 \times 0.6 \; + \; 0.4 \times 0.6 \times 0.3 = 0.18. \end{aligned}$$

$\mathrm{P}(Y_3 = 2 \mid Y_2 = 0) = p^{02}$

So

$$\mathbf{APV(FB)}_0 = 10^4\left(\frac{0.1}{1.06} + \frac{0.18}{(1.06)^2} + \frac{0.18}{(1.06)^3}\right) = 4056.70.$$

Finally $P = \frac{4056.70}{1.8864} = \mathbf{2150.5007}$.

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2. a. If the insured is healthy at time 1, then

$$
\begin{aligned}
{}_1V^{(0)} &= \mathbf{APV(FB)}_1^{(0)} - \mathbf{APV(FP)}_1^{(0)} \\
&= 10^4\left(\frac{\mathrm{P}\left(Y_2=2|Y_1=0\right)}{1.06}+\frac{\mathrm{P}\left(Y_2\neq 2, Y_3=2|Y_1=0\right)}{(1.06)^2}\right) - P\left(1+\frac{\mathrm{P}\left(Y_2=0|Y_1=0\right)}{1.06}\right)
\end{aligned}
$$

clearly $\mathrm{P}\left(Y_2=2|Y_1=0\right)=p^{02}=0.1$, $\mathrm{P}\left(Y_2=0|Y_1=0\right)=p^{00}=0.6$. and

$$
\begin{aligned}
\mathrm{P}\left(Y_2\neq 2, Y_3=2|Y_1=0\right) &= \mathrm{P}\left(Y_2=0, Y_3=2|Y_1=0\right)+\mathrm{P}\left(Y_2=1, Y_3=2|Y_1=0\right) \\
&= \mathrm{P}\left(Y_3=2|Y_2=0\right)\mathrm{P}\left(Y_2=0|Y_1=0\right) \\
&\quad +\mathrm{P}\left(Y_3=2|Y_2=1\right)\mathrm{P}\left(Y_2=1|Y_1=0\right) \\
&= p^{02}\,p^{00}+p^{12}\,p^{01}=0.1\times 0.6+0.4\times 0.3=0.18
\end{aligned}
$$

Consequently ${}_1V^{(0)}=10^4\left(\frac{0.1}{1.06}+\frac{0.18}{(1.06)^2}\right)-2150.5007\left(1+\frac{0.6}{1.06}\right)=\mathbf{-822.38}$.
Remember that the negative reserve corresponds to a gain of the insurance company.
b. If the insured is disabled at time 1, the premium is waived: that is there is no further premiums payable, thus $\mathbf{APV(FP)}_1^{(1)}=0$, therefore

$$
{}_1V^{(1)}=\mathbf{APV(FB)}_1^{(1)}-\mathbf{APV(FP)}_1^{(1)}=10^4\left(\frac{\mathrm{P}\left(Y_2=2|Y_1=1\right)}{1.06}+\frac{\mathrm{P}\left(Y_2\neq 2, Y_3=2|Y_1=1\right)}{(1.06)^2}\right)
$$

and clearly $\mathrm{P}\left(Y_2=2|Y_1=1\right)=p^{12}=0.4$ and

$$
\begin{aligned}
\mathrm{P}\left(Y_2\neq 2, Y_3=2|Y_1=1\right) &= \mathrm{P}\left(Y_2=1, Y_3=2|Y_1=1\right) \\
&= \mathrm{P}\left(Y_3=2|Y_2=1\right)\mathrm{P}\left(Y_2=1|Y_1=1\right) \\
&= p^{12}\,p^{11}=0.4\times 0.6=0.24
\end{aligned}
$$

Finally,

$$
{}_1V^{(1)}=10^4\left(\frac{0.4}{1.06}+\frac{0.24}{(1.06)^2}\right)=\mathbf{5909.5764}.
$$

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10.4 Continuous–time Markov Chain

Now we come to the construction of a multiple state model for which state changes can occur any time. Consider a life aged (x) at time $t=0$. For each t, the life can be in one (and only one) of the many states. We let $Y_u=i$ to mean that the individual is in state i at age u. Then $\{Y_{x+t}: t\geq 0\}$ would be a **continuous–time stochastic process** t. Let the state space of Y be E. As time evolves, Y is constant when there is no change in state and it would jump when there is a change in state. For the disability income model, suppose that we start with a healthy life aged x at time 0. So, the process starts with $Y_x=0$. It then undergoes a series of jumps between 0 and 1.

For a multiple state model, our first goal is to obtain the following probabilities:

- transition probability ${}_tp_u^{ij}=\mathrm{P}\left(Y_{u+t}=j \mid Y_u=i\right)$
- occupancy probability ${}_tp_u^{\overline{ii}}=\mathrm{P}\left(Y_{u+s}=i \text{ for } s\in[0,t] \mid Y_u=i\right)$

Note that ${}_tp_u^{ii}\geq {}_tp_u^{\overline{ii}}$, because the events that correspond to ${}_tp_u^{ii}$ include those that leaves state i, and then re–enter state i again on or before time t. Only when state i cannot be re–entered again

once it has been left (just like states 0 and 1 in the permanent disability income model) or when state i is absorbing would ${}_tp_u^{ii} = {}_tp_u^{\overline{ii}}$ hold.

We make the following assumptions for the multiple–state model, which is actually a non–homogeneous continuous–time Markov chain (**CTMC**):

Assumptions of continuous–time Markov chain

1. The Markov property:

$$\mathrm{P}\left(Y_{u+t} = j \mid Y_v, \text{ for } v \in [0,u]\right) = \mathrm{P}\left(Y_{u+t} = j \mid Y_u\right)$$

   Loosely speaking, the above means that the value of ${}_tp_u^{ij}$ does not depend on what has happened before the individual reaches age u. It only depends on the fact that the person is in state i at age u.

2. The probability for 2 or more transitions within a time period of length h is $o(h)$.

3. ${}_tp_u^{ij}$ is a differentiable function of t.

10.4.1 The Transition Intensity Matrix and Transition Probability Matrix

Similar to multiple decrement models in a CTMC the **inputs** are transition intensities. Suppose that there are $n+1$ possible states ($n > 0$) such that the state space $E = \{0, 1, 2, \ldots, n\}$. For $i, j \in E$ and $t \geq 0$

$$\begin{aligned}
\mu_{x+t}^{ij} &= \lim_{h \to 0} \frac{{}_hp_{x+t}^{ij}}{h} \quad \text{be the time–}t\text{ instantaneous rate of transition} \\
&\qquad\qquad \text{from state } i \text{ to state } j \text{ for } \mathbf{i \neq j}; \\
\nu_{x+t}^{i} &= \sum_{k=0, k\neq i}^{n} \mu_{x+t}^{ik} \quad \text{be the time–}t\text{ exit rate of state } i.
\end{aligned}$$

The transition rates and exit rates are then put into a **transition intensity matrix** (**Q**) :

$$\mathbf{Q}_{x+t} = \begin{pmatrix}
-\nu_{x+t}^{0} & \mu_{x+t}^{01} & \mu_{x+t}^{02} & \cdots & \cdots & \cdots & \cdots & \mu_{x+t}^{0n} \\
\mu_{x+t}^{10} & -\nu_{x+t}^{1} & \mu_{x+t}^{12} & \cdots & \cdots & \cdots & \cdots & \mu_{x+t}^{1n} \\
\mu_{x+t}^{20} & \mu_{x+t}^{21} & -\nu_{x+t}^{2} & \cdots & \cdots & \cdots & \cdots & \mu_{x+t}^{2n} \\
\vdots & \vdots & \vdots & & & & & \vdots \\
\vdots & \vdots & \vdots & & & \ddots & & \vdots \\
\vdots & \vdots & \vdots & & & & & \vdots \\
\vdots & \vdots & \vdots & & & & & \vdots \\
\mu_{x+t}^{n0} & \mu_{x+t}^{n1} & \mu_{x+t}^{n2} & \cdots & \cdots & \cdots & \cdots & -\nu_{x+t}^{n}
\end{pmatrix}$$

The sum of elements in any row of $\mathbf{Q}_{x+t}$ is **zero**. A state that cannot be left once it is entered is called an absorbing (or cemetery) state and the row that corresponds to it contains only 0.

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The transition probabilities ${}_tp_x^{ij}$ are entries of a **transition probability matrix (P)**:

$$ {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x^{00} & {}_tp_x^{01} & {}_tp_x^{02} & \cdots & \cdots & \cdots & \cdots & {}_tp_x^{0n} \\ {}_tp_x^{10} & {}_tp_x^{11} & {}_tp_x^{12} & \cdots & \cdots & \cdots & \cdots & {}_tp_x^{1n} \\ {}_tp_x^{20} & {}_tp_x^{21} & {}_tp_x^{22} & \cdots & \cdots & \cdots & \cdots & {}_tp_x^{2n} \\ \vdots & \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & & & \vdots \\ {}_tp_x^{n0} & {}_tp_x^{n1} & {}_tp_x^{n2} & \cdots & \cdots & \cdots & \cdots & {}_tp_x^{nn} \end{pmatrix} $$

The sum of elements in any row of ${}_t\mathbf{P}_x$ is 1. For an absorbing state, all elements in the row, except the one in the main diagonal, are 0. The one in the main diagonal is 1.

Now, let us formulate the three examples given in the introduction as CTMCs and see how their **Q** and **P** matrices look like.

10.4.2 Single Decrement Model

There are only 2 states: State 0 = (x) is alive and State 1 = (x) is dead (absorbing state), so $\mu_{x+t}^{01} = \mu_{x+t}$ and $\mu_{x+t}^{10} = 0$, ${}_tp_x^{00} = {}_tp_x$ and ${}_tp_x^{01} = {}_tq_x$, hence the transition intensity and probability matrices are

$$ \mathbf{Q}_{x+t} = \begin{pmatrix} -\mu_{x+t} & \mu_{x+t} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x & {}_tq_x \\ 0 & 1 \end{pmatrix} $$

10.4.3 Double Decrement Model

There are 3 states: State 0 = (x) is alive, State 1 = (x) dies from accident (absorbing state) and State 2 = (x) dies from other causes (absorbing state). Observe that transitions from state 0 to 1 and from state 0 to 2 are the only possible transitions. Thus so $\mu_{x+t}^{01} = \mu_{x+t}^{(1)}$, $\mu_{x+t}^{02} = \mu_{x+t}^{(2)}$ then $\nu^0 = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)} = \mu_{x+t}^{(\tau)}$, $\mu_{x+t}^{10} = \mu_{x+t}^{12} = 0$, thus $\nu^1 = 0$ and $\mu_{x+t}^{20} = \mu_{x+t}^{21} = 0$, thus $\nu^2 = 0$. And ${}_tp_x^{01} = {}_tq_x^{(1)}$, ${}_tp_x^{02} = {}_tq_x^{(2)}$, thus ${}_tp_x^{00} = {}_tp_x^{(\tau)}$ also ${}_tp_x^{11} = {}_tp_x^{22} = 1$, hence ${}_tp_x^{10} = {}_tp_x^{12} = {}_tp_x^{20} = {}_tp_x^{21} = 0$, the transition intensity and probability matrices are

$$ \mathbf{Q}_{x+t} = \begin{pmatrix} -\mu_{x+t}^{(\tau)} & \mu_{x+t}^{(1)} & \mu_{x+t}^{(2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x^{(\tau)} & {}_tq_x^{(1)} & {}_tq_x^{(2)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

The primed rates, being unobservable rates, cannot be expressed as functions of ${}_tp_x^{ij}$. So we need

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to define new symbols for them. The corresponding symbols are given in the following table:

Models	**Standard**	**Multiple state**
Force of decrement	$\mu_{x+t}^{(i)}$	μ_{x+t}^{0i}
Total force of decrement	$\mu_{x+t}^{(\tau)}$	$\mu_{x+t}^{0\cdot} = \nu_{x+t}^{0}$
Survival probability	${}_tp_x^{(\tau)}$	${}_tp_x^{00}$
Transition probability	${}_tq_x^{(i)}$	${}_tp_x^{0i}$
Total transition probability	${}_tq_x^{(\tau)}$	${}_tp_x^{0\cdot}$
Independent survival probability	${}_tp_x'^{(i)}$	${}_tp_x^{i}$
Absolute rate of decrement	${}_tq_x'^{(i)}$	${}_tq_x^{i}$
Remember that $\nu_{x+t}^{i} = \sum_{k=0,k\neq i}^{n} \mu_{x+t}^{ik}$		

Example 10.4.1 *For a three–state model with states numbered* 0, 1, *and* 2, *you are given: (i) The only possible transitions are* 0 *to* 1 *and* 0 *to* 2. *(ii)* $\mu_{x+t}^{01} = 0.3$ *for* $t \geq 0$ *and* $\mu_{x+t}^{02} = 0.4$ *for* $t \geq 0$. *Calculate* ${}_2p_{x+1}^{0\cdot}$ *and* ${}_2p_{x+1}^{02}$.

Solution: Observe first that this multiple state model is equivalent to a double decrement model with $\mu_{x+t}^{(1)} = 0.3$ for $t \geq 0$ and $\mu_{x+t}^{(2)} = 0.4$, according to the above symbols we are asked to compute ${}_2q_{x+1}^{(\tau)}$ and ${}_2q_{x+1}^{(2)}$. Since ${}_2q_{x+1}^{(2)} = \int_0^2 {}_sp_{x+1}^{(\tau)}\, \mu_{x+1+s}^{(2)}\, \mathrm{d}s$, let us first compute ${}_sp_{x+1}^{(\tau)}$, we have ${}_sp_{x+1}^{(\tau)} = e^{-\mu^{(\tau)}s} = e^{-0.7s}$, thus

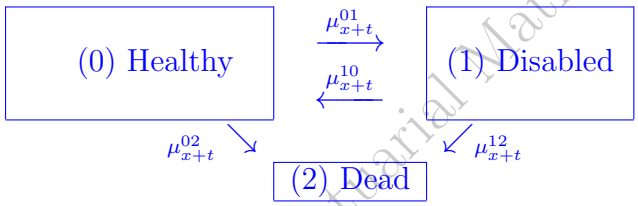
$$ {}_2q_{x+1}^{(\tau)} = 1 - {}_2p_{x+1}^{(\tau)} = 1 - e^{-2\times 0.7} = \mathbf{0.7534} $$

and

$$ {}_2q_{x+1}^{(2)} = \int_0^2 e^{-0.7s} 0.4\, \mathrm{d}s = \frac{4}{7}\left(1 - e^{-2\times 0.7}\right) = \frac{4}{7}(0.7534) = \mathbf{0.43051}. $$

10.4.4 Disability Income Model

Suppose the various rates of transition are labeled as follows:



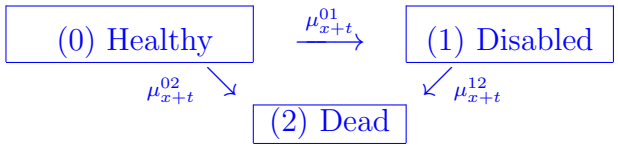
The state 2 is an absorbing state. The transition intensity and probability matrices are

$$ \mathbf{Q}_{x+t} = \begin{pmatrix} -\left(\mu_{x+t}^{01} + \mu_{x+t}^{02}\right) & \mu_{x+t}^{01} & \mu_{x+t}^{02} \\ \mu_{x+t}^{10} & -\left(\mu_{x+t}^{10} + \mu_{x+t}^{12}\right) & \mu_{x+t}^{12} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x^{00} & {}_tp_x^{01} & {}_tp_x^{02} \\ {}_tp_x^{10} & {}_tp_x^{11} & {}_tp_x^{12} \\ 0 & 0 & 1 \end{pmatrix} $$

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10.4.5 Permanent Disability Model

The diagram for this model is as follows



The permanent disability model can be obtained by setting $\mu^{10}_{x+t} = 0$. The transition intensity and probability matrices for this model are

$$\mathbf{Q}_{x+t} = \begin{pmatrix} -\left(\mu^{01}_{x+t} + \mu^{02}_{x+t}\right) & \mu^{01}_{x+t} & \mu^{02}_{x+t} \\ 0 & -\mu^{12}_{x+t} & \mu^{12}_{x+t} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp^{00}_x & {}_tp^{01}_x & {}_tp^{02}_x \\ 0 & {}_tp^{11}_x & {}_tp^{12}_x \\ 0 & 0 & 1 \end{pmatrix}$$

Similar to a discrete–time Markov chain, the CK equation holds in CTMC

$${}_{t+s}\mathbf{P}_x = {}_t\mathbf{P}_x \; {}_s\mathbf{P}_{x+t} \quad \text{for all} \quad x,\ s,\ t \ \text{ in } \mathbb{R}_+.$$

10.5 Kolmogorov's Forward Equations

We first study how the transition intensities govern the local behavior of transition probabilities. Then we would see how $\mathbf{P}$ can be obtained from $\mathbf{Q}$.

Component form $$\frac{\mathrm{d}\left({}_tp^{ij}_x\right)}{\mathrm{d}t} = \sum_{k=0,k\neq j}^{n} {}_tp^{ik}_x \mu^{kj}_{x+t} - {}_tp^{ij}_x \; \nu^{j}_{x+t}$$

with initial conditions ${}_0p^{ii}_x = 1$ and ${}_0p^{ij}_x = 0$ for $i \neq j$.

Matrix form $$\frac{\mathrm{d}\left({}_t\mathbf{P}_x\right)}{\mathrm{d}t} = {}_t\mathbf{P}_x \; \mathbf{Q}_{x+t}$$ with initial condition ${}_0\mathbf{P}_x = \mathbf{I}$ (the identity matrix)

10.5.1 Applications

Example 10.5.1 *Consider the* ***single decrement model****. Use KFE to derive the expression for* ${}_tp_x$ *and* ${}_tq_x$

Solution: Remember that the transition intensity and probability matrices are

$$\mathbf{Q}_{x+t} = \begin{pmatrix} -\mu_{x+t} & \mu_{x+t} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x & {}_tq_x \\ 0 & 1 \end{pmatrix}$$

Foe from the KFE we can write

$$\frac{d}{\mathrm{d}t}\begin{pmatrix} {}_tp_x & {}_tq_x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} {}_tp_x & {}_tq_x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\mu_{x+t} & \mu_{x+t} \\ 0 & 0 \end{pmatrix}$$

which is equivalent to $\frac{\mathrm{d}\ {}_tp_x}{\mathrm{d}t} = -\ {}_tp_x\mu_{x+t}$ and $\frac{\mathrm{d}\ {}_tq_x}{\mathrm{d}t} = {}_tp_x\mu_{x+t}$ (i.e. therefore $f_x(t) = {}_tp_x\mu_{x+t}$). The solution to the linear equation $\frac{\mathrm{d}\ {}_tp_x}{\mathrm{d}t} = -\ {}_tp_x\mu_{x+t}$ with ${}_0p_x = 1$ is given ${}_tp^{00}_x = {}_tp_x = e^{-\int_0^t \mu_{x+u}\ \mathrm{d}u}$ and ${}_tq_x - {}_0q_x = \int_0^t {}_sp_x\mu_{x+s}\ \mathrm{d}s$ with ${}_0q_x = 0$.

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Example 10.5.2 *Consider a multiple decrement model with n decrements. Use KFE to analyze the model and derive the expressions for* ${}_tp_x^{(\tau)}$ *and* ${}_tq_x^{(i)}$.

Solution: Similar to a double decrement model, the transition intensity and probability matrices are

$$\mathbf{Q}_{x+t} = \begin{pmatrix} -\mu_{x+t}^{(\tau)} & \mu_{x+t}^{(1)} & \mu_{x+t}^{(2)} & \cdots & \cdots & \cdots & \mu_{x+t}^{(n)} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \text{ and } {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp_x^{(\tau)} & {}_tq_x^{(1)} & {}_tq_x^{(2)} & \cdots & \cdots & \cdots & {}_tq_x^{(n)} \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

KFE says that

$$\frac{d}{\mathrm{d}t} \begin{pmatrix} {}_tp_x^{(\tau)} & {}_tq_x^{(1)} & {}_tq_x^{(2)} & \cdots & \cdots & \cdots & {}_tq_x^{(n)} \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} {}_tp_x^{(\tau)} & {}_tq_x^{(1)} & {}_tq_x^{(2)} & \cdots & \cdots & \cdots & {}_tq_x^{(n)} \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} -\mu_{x+t}^{(\tau)} & \mu_{x+t}^{(1)} & \mu_{x+t}^{(2)} & \cdots & \cdots & \cdots & \mu_{x+t}^{(n)} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

with initial conditions ${}_0p_x^{(\tau)} = 1$ and ${}_0q_x^{(i)} = 0$ for all $i = 1, 2, \ldots, n$.

The component form for the $(00)^{th}$ element is

$$\frac{d_t p_x^{(\tau)}}{\mathrm{d}t} = -\,{}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}$$

which gives as before

$${}_tp_x^{(\tau)} = e^{-\int_0^t \mu_{x+u}^{(\tau)}\,\mathrm{d}u}$$

The component form for the $(0j)^{th}$ element $(j > 0)$ is

$$\frac{d_t q_x^{(j)}}{\mathrm{d}t} = {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}$$

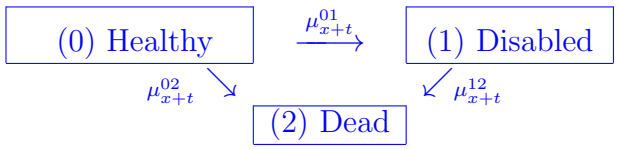
which gives by integrating both sides

$${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\mu_{x+s}^{(j)}\,\mathrm{d}s \quad \text{since} \quad {}_0q_x^{(j)} = 0$$

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The Permanent Disability Model

Armed with KFE and the analytic solution to occupancy probability, we can finally tackle the permanent disability model analytically.



The permanent disability model can be obtained by setting $\mu^{10}_{x+t} = 0$. The transition intensity and probability matrices for this model are

$$\mathbf{Q}_{x+t} = \begin{pmatrix} -\left(\mu^{01}_{x+t} + \mu^{02}_{x+t}\right) & \mu^{01}_{x+t} & \mu^{02}_{x+t} \\ 0 & -\mu^{12}_{x+t} & \mu^{12}_{x+t} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad {}_t\mathbf{P}_x = \begin{pmatrix} {}_tp^{00}_x & {}_tp^{01}_x & {}_tp^{02}_x \\ 0 & {}_tp^{11}_x & {}_tp^{12}_x \\ 0 & 0 & 1 \end{pmatrix}$$

Using again the KFE

$$\frac{d}{dt}\begin{pmatrix} {}_tp^{00}_x & {}_tp^{01}_x & {}_tp^{02}_x \\ 0 & {}_tp^{11}_x & {}_tp^{12}_x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}_tp^{00}_x & {}_tp^{01}_x & {}_tp^{02}_x \\ 0 & {}_tp^{11}_x & {}_tp^{12}_x \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\left(\mu^{01}_{x+t} + \mu^{02}_{x+t}\right) & \mu^{01}_{x+t} & \mu^{02}_{x+t} \\ 0 & -\mu^{12}_{x+t} & \mu^{12}_{x+t} \\ 0 & 0 & 1 \end{pmatrix}$$

which leads to

$$ {}_tp^{00}_x = e^{-\int_0^t \nu^0_{x+u}\,du} = e^{-\int_0^t (\mu^{01}_{x+u}+\mu^{02}_{x+u})\,du}, \quad {}_tp^{11}_x = e^{-\int_0^t \mu^{12}_{x+u}\,du}, \tag{10.1}$$

$$ {}_tp^{01}_x = \int_0^t {}_sp^{00}_x\,\mu^{01}_{x+s}\,{}_{t-s}p^{11}_{x+s}\,ds, \quad {}_tp^{00}_x + {}_tp^{01}_x + {}_tp^{02}_x = 1 \text{ and } {}_tp^{11}_x + {}_tp^{12}_x = 1 \tag{10.2}$$

Example 10.5.3 *Calculate ${}_{10}p^{11}_{60}$, ${}_{10}p^{12}_{60}$, ${}_{10}p^{00}_{60}$, ${}_{10}p^{01}_{60}$ and ${}_{10}p^{02}_{60}$ for the permanent disability model if*
(a) $\mu^{01}_x = 0.028$, $\mu^{02}_x = 0.023$ and $\mu^{12}_x = 0.032$.
(b) $\mu^{01}_x = 4 \times 10^{-4} + 3.5 \times 10^{-6} e^{0.15x}$, $\mu^{02}_x = \mu^{12}_x = 8 \times 10^{-5} e^{0.1x}$. For (b), use Simpson's rule with 4 subintervals.

Solution: (a) In this example all transition intensities are constant. The matrix $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{pmatrix} -0.051 & 0.028 & 0.023 \\ 0 & -0.032 & 0.032 \\ 0 & 0 & 0 \end{pmatrix}$$

By equations (10.1) and (10.2) we have ${}_tp^{00}_{60} = e^{-\int_0^t 0.051\,du} = e^{-0.051t}$, so ${}_{10}p^{00}_{60} = e^{-0.51} = 0.60050$ and ${}_tp^{11}_{60} = e^{-\int_0^t 0.032\,du} = e^{-0.032t}$ thus ${}_{10}p^{11}_{60} = e^{-0.32} = 0.72615$ and ${}_{10}p^{12}_{60} = 1 - 0.72615 = 0.27385$. Moreover

$${}_tp^{01}_x = \int_0^t e^{-0.051s} 0.028\, e^{-0.032(t-s)}\,ds = 0.028\, e^{-0.032t}\int_0^t e^{-0.019s}\,ds = \frac{28}{19} e^{-0.032t}\left(1 - e^{-0.019t}\right)$$

$${}_{10}p^{01}_{60} = \frac{28}{19} e^{-0.32}\left(1 - e^{-0.19}\right) = 0.18517.$$

and

$${}_{10}p^{02}_{60} = 1 - {}_{10}p^{00}_{60} - {}_{10}p^{01}_{60} = 1 - 0.60050 - 0.18517 = 0.21433.$$

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(b) Again (10.2) we have

$$ {}_{10}p^{01}_{60} = \int_0^{10} {}_sp^{00}_{60}\ \mu^{01}_{60+s}\ {}_{10-s}p^{11}_{60+s}\ \mathrm{d}s $$

and

$$ \begin{aligned} {}_sp^{00}_{60} &= \exp\left(-\int_0^s \left(4\times10^{-4} + 3.5\times10^{-6}e^{0.1(60+u)} + 8\times10^{-5}e^{0.1(60+u)}\right)\ \mathrm{d}u\right) \\ &= \exp\left(-4\times10^{-4}s - \frac{3.5\times10^{-6}e^{9}}{0.15}\left(e^{0.15s}-1\right) - \frac{8\times10^{-5}e^{6}}{0.1}\left(e^{0.1s}-1\right)\right) \\ &= \exp\left(0.51181 - 0.18907e^{0.15s} - 0.32274e^{0.1s} - 0.0004s\right). \end{aligned} $$

As a by product, putting $s = 10$, we get

$$ {}_{10}p^{00}_{60} = \exp\left(-4\times10^{-3} - \frac{3.5\times10^{-6}e^{9}}{0.15}\left(e^{1.5}-1\right) - \frac{8\times10^{-5}e^{6}}{0.1}\left(e-1\right)\right) = 0.29616. $$

and

$$ \begin{aligned} {}_{10-s}p^{11}_{60+s} &= \exp\left(-\int_0^{10-s} 8\times10^{-5}e^{0.1(60+s+u)}\ \mathrm{d}u\right) \\ &= \exp\left(-\frac{8\times10^{-5}e^{6+0.1s}}{0.1}\left(e^{0.1(10-s)}-1\right)\right) \\ &= \exp\left(-8\times10^{-4}e^{6}\left(e-e^{0.1s}\right)\right) = \exp\left(0.32274e^{0.1s} - 0.87731\right) \end{aligned} $$

Hence ${}_{10}p^{11}_{60} = \exp\left(0.32274 - 0.87731\right) = 0.57432$ and ${}_{10}p^{12}_{60} = 1 - 0.57432 = 0.42568.$

By using the two expressions, we want to compute $\int_0^{10} f(s)\ \mathrm{d}s$ where

$$ \begin{aligned} f(s) &= {}_sp^{00}_{60}\ \mu^{01}_{60+s}\ {}_{10-s}p^{11}_{60+s} \\ &= \exp\left(-4\times10^{-4}s - \frac{3.5\times10^{-6}e^{9}}{0.15}\left(e^{0.15s}-1\right) - \frac{8\times10^{-5}e^{6}}{0.1}\left(e^{0.1s}-1\right)\right) \\ &\quad\times 4\times10^{-4} + 3.5\times10^{-6}e^{0.15(60+s)} \times \exp\left(-8\times10^{-4}e^{6}\left(e-e^{0.1s}\right)\right) \end{aligned} $$

at $s = 0,\ 2.5,\ 5,\ 7.5$ and 10 $(h = \frac{10}{4} = 2.5)$

s	${}_sp^{00}_{60}$	μ^{01}_{60+s}	${}_{10-s}p^{11}_{60+s}$	Integrand $f(s)$
0	1.000000	0.028761	0.574319	0.016518
2.5	0.836364	0.041665	0.629453	0.021934
5	0.655367	0.060440	0.708078	0.028047
7.5	0.469162	0.087757	0.823601	0.033910
10	0.296164	0.127504	0.999988	0.037762

Therefore Simpson's rule

$$ \int_a^b f(x)dx \simeq \frac{h}{3}\left\{f(a) + 4\left[f(x_1) + f(x_3) + \cdots + f(x_{2n-1})\right] + 2\left[f(x_2) + f(x_4) + \cdots + f(x_{2n-2})\right] + f(b)\right\} $$

with $h = 2.5$ gives

$$ {}_{10}p^{01}_{60} \simeq \frac{2.5}{3}\left(0.016518 + 4\left(0.021934 + 0.033910\right) + 2\times0.028047 + 0.037762\right) = \mathbf{0.278135}. $$

and ${}_{10}p^{02}_{60} = 1 - {}_{10}p^{00}_{60} - {}_{10}p^{01}_{60} = 1 - 0.29616 - 0.278135 \simeq 0.42571$

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10.6 Calculating Actuarial Present Value of Cash Flows

In a multiple state model, cash flows can arise from many different situations. For example, in a permanent disability insurance plan, some or all of the following may be provided:

- an annuity while the insured is disabled (state 1)
- a lump sum at the moment of becoming disable (transition from state 0 to state 1)
- a lump sum at the moment of death (transition into state 2)

In general, we have the following cash flows in a multiple state model:

10.6.1 Benefits cash flows

(1) Cash flow arising from a specific transition (e.g. transition from state 1 to state 2)

(2) Cash flow arising from leaving a specific state (e.g. every time when Y leaves state 0)

(3) Cash flows that arise when entering a specific state (e.g. every time when Y enters state 1)

10.6.2 Premium cash flows

(4) Cash flows that arise when Y is in a specific state (e.g. when Y is state 0 in the first year or at the beginning of every year).

In what follows, assume that the life age (x) is in state i at time 0.

(i) Suppose that b_t is payable at time t if a transition from state k to state j happens at time t for $t \leq m$. The probability to get b_t in $(t, t+\mathrm{d}t)$ is the probability that there is a transition from state k to state j in $(t, t+\mathrm{d}t)$ which is ${}_tp_x^{ik}\ \mu_{x+t}^{kj}\ \mathrm{d}t$. So the APV $\int_0^m b_t\ e^{-\delta t}\ {}_tp_x^{ik}\ \mu_{x+t}^{kj}\ \mathrm{d}t$.

(ii) Suppose that b_t is payable at time t if Y leaves state k at time t for $t \leq m$, then the APV is

$$\sum_{j\neq k}\int_0^m b_t\ e^{-\delta t}\ {}_tp_x^{ik}\ \mu_{x+t}^{kj}\ \mathrm{d}t = \int_0^m b_t\ e^{-\delta t}\ {}_tp_x^{ik}\ \nu_{x+t}^{k}\ \mathrm{d}t.$$

(iii) Suppose that b_t is payable at time t if Y enters the state j at time t for $t \leq m$, then the APV is

$$\sum_{k\neq j}\int_0^m b_t\ e^{-\delta t}\ {}_tp_x^{ik}\ \mu_{x+t}^{kj}\ \mathrm{d}t = \int_0^m b_t\ e^{-\delta t}\ \sum_{k\neq j}\ {}_tp_x^{ik}\ \mu_{x+t}^{kj}\ \mathrm{d}t.$$

If $b_t = 1$ then one can use the following notation

$$\bar{A}^{ij}_{x:\overline{m}|} = \int_0^m e^{-\delta t}\ \sum_{k\neq j}\ {}_tp_x^{ik}\ \mu_{x+t}^{kj}\ \mathrm{d}t$$

For discrete insurance, paying a benefit at the end of the year Y enters the state j, we shall denote

$$A^{ij}_{x:\overline{m}|} = \sum_{\ell=0}^{m-1} v^{\ell+1}\left(\sum_{k\neq j}\ {}_\ell p_x^{ik}\ p_{x+\ell}^{kj}\right).$$

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(iv) Suppose there is an annuity of 1 per year payable continuously for m years while the life is in state j. The APV of this annuity is, assuming a constant interest rate, is

$$\bar{a}^{ij}_{x:\overline{m}|} = \int_0^m e^{-\delta t}\, {}_tp^{ij}_x \, \mathrm{d}t$$

Similarly, if the annuity is payable at the beginning of each year, we have

$$\ddot{a}^{ij}_{x:\overline{m}|} = \sum_{k=0}^{m-1} v^k \, {}_kp^{ij}_x.$$

10.6.3 Practice

Example 10.6.1 *Consider the permanent disability model with transition intensities*

$$\mu^{01}_x = 0.028, \quad \mu^{02}_x = 0.023 \quad \textit{and} \quad \mu^{12}_x = 0.032.$$

Assume that the effective interest rate is 5% per year. For a life age (x) that is healthy, calculate the actuarial present value of

1. *an insurance that pays* 1000 *at the moment when the insured becomes disabled;*
2. *an annuity due that pays* 10 *at the beginning of each month if the insured is healthy, for at most* 10 *years;*
3. *an annuity due that pays* 100 *at the beginning of each month if the insured is disabled, for at most* 3 *years starting from the time when the first annuity payment is made.*

Solution:

1. The density of the time to transit from state 0 to state 1 is ${}_tp^{00}_x \ \mu^{01}_{x+t} = 0.028e^{-0.051t}$. The actuarial present value of the insurance is

$$1000 \int_0^\infty e^{-\ln(1.05)t}\, 0.028e^{-0.051t}\, \mathrm{d}t = 28 \int_0^\infty e^{-(\ln(1.05)+0.051)t}\, \mathrm{d}t = \frac{28}{\ln(1.05)+0.051} = 280.5892374$$

2. The actuarial present value of the annuity is

$$10 \times 12\ddot{a}^{(12)}_{x:\overline{10}|} = 10 \sum_{k=0}^{10\times 12-1} v^{\frac{k}{12}}\, {}_{\frac{k}{12}}p^{00}_x = 10 \sum_{k=0}^{119} \left(ve^{-0.051}\right)^{\frac{k}{12}} = 10\frac{1-(0.99172)^{120}}{1-0.99172} = 762.4237515.$$

3. The maximum number of annuity payment is 36. Let us fix a time point t (which is a multiple of $\frac{1}{12}$). For 100 to be payable at t, the insured has to be in state 1, but not in state 1 **for more than 3 years**.

   (i) $t < 3$. As a result, the probability of a payment at t is, for $t = 0, \ \frac{1}{12}, \ \frac{2}{12}, \ \ldots, \ \frac{35}{12}$, is

$$\begin{aligned} \mathrm{P}\left(Y_{x+t} = 1 \mid Y_x = 0\right) &= {}_tp^{01}_x = \int_0^t e^{-0.051s} 0.028\, e^{-0.032(t-s)}\, \mathrm{d}s \\ &= 0.028\, e^{-0.032t} \int_0^t e^{-0.019s}\, \mathrm{d}s = \frac{28}{19} e^{-0.032t}\left(1 - e^{-0.019t}\right) \end{aligned}$$

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(ii) $t \geq 3$, the probability of a payment at t is

$$\begin{aligned} \mathrm{P}\left(Y_{x+t}=1,\ Y_{x+t-3}=0 \mid Y_x=0\right) &= \mathrm{P}\left(Y_{x+t-3}=0 \mid Y_x=0\right)\mathrm{P}\left(Y_{x+t}=1 \mid Y_{x+t-3}=0\right) \\ &= {}_{t-3}p_x^{00}\ {}_3p_{x+t-3}^{01} = {}_{t-3}p_x^{00}\ {}_3p_x^{01}, \end{aligned}$$

since all transition rates are independent of age. Moreover

$$e^{-0.051(t-3)}\ {}_3p_x^{01} = e^{-0.051(t-3)}\frac{28}{19}e^{-0.032\times 3}\left(1-e^{-0.019\times 3}\right)$$

If κ = APV of the annuity, then

$$\begin{aligned} \kappa &= 100\sum_{k=0}^{10\times 12-1} v^{\frac{k}{12}}\mathrm{P}\left(Y_{x+\frac{k}{12}}=1 \mid Y_x=0\right) + 100\sum_{k=36}^{\infty} v^{\frac{k}{12}}\mathrm{P}\left(Y_{x+\frac{k}{12}}=1,\ Y_{x+\frac{k}{12}-3}=0 \mid Y_x=0\right) \\ &= 100\sum_{k=0}^{10\times 12-1} v^{\frac{k}{12}}\ {}_{\frac{k}{12}}p_x^{01} + 100\ {}_3p_x^{01}\sum_{k=36}^{\infty} v^{\frac{k}{12}}\ {}_{\frac{k}{12}-3}p_x^{00} \\ &= 100\sum_{k=0}^{35} v^{\frac{k}{12}}\ {}_{\frac{k}{12}}p_x^{01} + 100\ {}_3p_x^{01}\sum_{k=36}^{\infty} v^{\frac{k}{12}}\ e^{-0.051(\frac{k}{12}-3)} \\ &= 100\sum_{k=0}^{35} v^{\frac{k}{12}}\frac{28}{19}e^{-0.032\frac{k}{12}}\left(1-e^{-0.019\frac{k}{12}}\right) + (100)\,e^{0.051(3)}\ {}_3p_x^{01}\sum_{k=36}^{\infty}\left(\left(1.05e^{0.051}\right)^{-\frac{1}{12}}\right)^k \\ &= \frac{2800}{19}\sum_{k=0}^{35}(1.05)^{-\frac{k}{12}}e^{-0.032\frac{k}{12}}\left(1-e^{-0.019\frac{k}{12}}\right) \\ &\quad + (100)\,e^{0.051(3)}\left(\left(\frac{28}{19}\right)e^{-0.032\times 3}\left(1-e^{-0.019\times 3}\right)\right)\sum_{k=36}^{\infty}\left(\left(1.05e^{0.051}\right)^{-\frac{1}{12}}\right)^k. \end{aligned}$$

Therefore

$$\begin{aligned} \kappa &= \frac{2800}{19}\sum_{k=0}^{35}\left((1.05)^{\frac{1}{12}}\times e^{\frac{0.032}{12}}\right)^{-k}\left(1-e^{-0.019\frac{k}{12}}\right) \\ &\quad + (100)\,e^{0.051(3)}\left(\left(\frac{28}{19}\right)e^{-0.032\times 3}\left(1-e^{-0.019\times 3}\right)\right)\sum_{k=36}^{\infty}\left(\left(1.05e^{0.051}\right)^{-\frac{1}{12}}\right)^k \\ &= \frac{2800}{19}\times 0.83652 + 8.644\times 89.51 = \mathbf{897.022}. \end{aligned}$$

where we have used

$$\sum_{k=0}^{35}\left((1.05)^{\frac{1}{12}}\times e^{\frac{0.032}{12}}\right)^{-k}\left(1-e^{-0.019\frac{k}{12}}\right) = 0.83652$$

and

$$\sum_{k=36}^{\infty}\left(\left(1.05e^{0.051}\right)^{-\frac{1}{12}}\right)^k = 89.51.$$

Example 10.6.2 *Consider a permanent disability model with three states: State 0: Healthy, State 1: Permanently disabled, and State 2: Dead. Suppose that* $\mu_x^{01}=0.02$, $\mu_x^{02}=0.04$, $\mu_x^{12}=0.05$ *for* $x \geq 0$. *For a person who is healthy at age* 50,

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1. *Calculate the probability that*

   (a) *he is healthy at age* 60*;*

   (b) *he cannot survive to age* 60*.*

2. *In a permanent disability model,* $\mu_x^{01} = 0.05$, $\mu_x^{02} = 0.02$, $\mu_x^{12} = 0.03$ *and* $\delta = 0.06$.

   (a) *Calculate* $\bar{a}_x^{00}$, $\bar{a}_x^{11}$ *and* $\bar{a}_x^{01}$.

   (b) *Calculate* $\bar{A}_x^{12}$ *and* $\bar{A}_x^{02}$.

3. *A permanent disability insurance pays continuously at the rate of* 5000 *per year while the insured is disabled. The policyholder pays continuous premiums while he is healthy.*

   (a) *Calculate the annual premium rate using the equivalence principle.*

   (b) *Calculate the continuous premium payable annually for an insurance paying a benefit* 8000 *when the life aged* (x) *enters the state* 2*, using the equivalence principle if:*

      i. *The premiums are payable in state* 0 *only.*

      ii. *The premiums are payable in both states* 0 *and* 1*.*

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Solution:

1.

(a) The total rate of exit from state 0 is $\nu_x^0 = \mu_x^{01} + \mu_x^{02} = 0.06$. We know that ${}_tp_{50}^{00} = e^{-0.06t}$, thus ${}_{10}p_{50}^{00} = e^{-0.6} = \mathbf{0.54881}$.

(b) The probability that a person who is healthy at age 50 cannot survive to age 60 is given by

$$1 - {}_{10}p_{50}^{00} - {}_{10}p_{50}^{01}.$$

So we need to calculate ${}_{10}p_{50}^{01}$. Indeed the total rate of exit from state 1 is 0.05. Similarly, ${}_tp_x^{11} = e^{-0.05t}$ for any $x \geq 0$ and $t \geq 0$. Now, from the formula sheet we have

$$\begin{aligned} {}_{10}p_{50}^{01} &= \int_0^{10} {}_sp_{50}^{00}\, \mu_{50+s}^{01}\, {}_{10-s}p_{50+s}^{11}\, \mathrm{d}s = \int_0^{10} e^{-0.06s}\, 0.02\, e^{-0.05(10-s)}\, \mathrm{d}s \\ &= 0.02e^{-0.5} \int_0^{10} e^{-0.01s}\, \mathrm{d}s = \frac{0.02e^{-0.5}}{0.01}\left(1 - e^{-0.1}\right) = 2e^{-0.5}\left(1 - e^{-0.1}\right). \end{aligned}$$

So the probability that a person who is healthy at age 50 cannot survive to age 60 is given by

$$1 - {}_{10}p_{50}^{00} - {}_{10}p_{50}^{01} = 1 - e^{-0.6} - 2e^{-0.5}(1 - e^{-0.1}) = \mathbf{0.33575}.$$

2.

(a) The first annuity is like a single–decrement annuity with a decrement rate of $0.05 + 0.02 = 0.07$, so its APV is

$$\bar{a}_x^{00} = \int_0^{\infty} e^{-\delta t}\, {}_tp_x^{00}\, \mathrm{d}t = \int_0^{\infty} e^{-\delta t} e^{-\nu^0 t}\, \mathrm{d}t = \frac{1}{\mu_x^{01} + \mu_x^{02} + \delta} = \frac{1}{0.13} = \mathbf{7.6923}.$$

Similarly, the second annuity has decrement rate $\mu^{12} = 0.03$ and has APV equal to

$$\bar{a}_x^{11} = \int_0^\infty e^{-\delta t}\, {}_tp_x^{11}\, \mathrm{d}t = \int_0^\infty e^{-\delta t} e^{-\mu^{12} t}\, \mathrm{d}t = \frac{1}{0.03+0.06} = \frac{100}{9} = \mathbf{11.1111}.$$

For the third annuity

$$\begin{aligned} {}_tp_x^{01} &= \int_0^t {}_sp_x^{00}\mu_{x+s}^{01}\, {}_{t-s}p_{x+s}^{11}\, \mathrm{d}s = 0.05\int_0^t e^{-0.07s}e^{-0.03(t-s)}\, \mathrm{d}s = 0.05e^{-0.03t}\int_0^t e^{-0.04s}\, \mathrm{d}s \\ &= \frac{0.05}{0.04}e^{-0.03t}\left(1-e^{-0.04t}\right) = \frac{5}{4}\left(e^{-0.03t}-e^{-0.07t}\right). \end{aligned}$$

Thus

$$\begin{aligned} \bar{a}_x^{01} &= \int_0^\infty e^{-\delta t}\, {}_tp_x^{01}\, \mathrm{d}t = \frac{5}{4}\int_0^\infty e^{-0.06t}\left(e^{-0.03t}-e^{-0.07t}\right)\, \mathrm{d}t = \frac{5}{4}\left(\int_0^\infty e^{-0.09t}\, \mathrm{d}t - \int_0^\infty e^{-0.13t}\, \mathrm{d}t\right) \\ &= \frac{5}{4}\left(\frac{1}{0.09}-\frac{1}{0.13}\right) = \mathbf{4.27350}. \end{aligned}$$

(b) $\bar{A}_x^{12}$ is a standard whole life insurance since there's only one path from state 1 to state 2. So it is $\frac{\mu^{12}}{\mu^{12}+\delta} = \frac{0.03}{0.03+0.06} = \frac{1}{3} = \mathbf{0.33333}$.
For $\bar{A}_x^{02}$ there are two paths from state 0 to state 2. The APV of the direct path is

$$\begin{aligned} \int_0^\infty e^{-\delta t}\, {}_tp_x^{00}\mu^{02}\, \mathrm{d}t &= \int_0^\infty e^{-\delta t}e^{-\nu^0 t}\mu^{02}\, \mathrm{d}t = \frac{\mu^{02}}{\nu^0+\delta} = \frac{\mu^{02}}{\mu_x^{01}+\mu_x^{02}+\delta} \\ &= \frac{0.02}{0.05+0.02+0.06} = \frac{2}{13} = 0.15385. \end{aligned}$$

For the other path, we use the formula for ${}_tp_x^{01}$ that we developed previously. So the APV of the path $(0 \to 1 \to 2)$ is

$$\int_0^\infty e^{-\delta t}\, {}_tp_x^{01}\mu^{12}\, \mathrm{d}t = 0.03\bar{a}_x^{01} = 0.03 \times 4.27350 = 0.12821.$$

Thus $\bar{A}_x^{02} = 0.15385 + 0.12821 = \mathbf{0.28206}$.

3.

(a) By the equivalence principle, $P\bar{a}_x^{00} = 5000\bar{a}_x^{01}$, hence $P = 5000\frac{4.2735}{7.6923} = \mathbf{2777.7778}$.

(b)

i. For an annuity in state 0, the APV(FP) is $P\bar{a}_x^{00} = 7.6923P$, and APV(FB) is $8000\bar{A}_x^{02}$ so the premium is

$$P = \frac{8000\bar{A}_x^{02}}{\bar{a}_x^{00}} = 8000\frac{0.282051}{7.6923} = \mathbf{293.3333}.$$

ii. For an annuity payable in state 1 for someone currently in state 0, the APV is 4.2735. So the APV of an annuity payable whether in state 0 or 1 is the sum of an annuity payable in state 0 and an annuity payable in state 1, APV(FP)= $P(\bar{a}_x^{00}+\bar{a}_x^{01})$ and APV(FB) is $8000\bar{A}_x^{02}$, hence premium for the insurance is

$$P = \frac{8000\bar{A}_x^{02}}{\bar{a}_x^{00}+\bar{a}_x^{01}} = 8000\frac{0.282051}{7.6923+4.2735} = \mathbf{188.5714}.$$

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Example 10.6.3 *Consider the disability income model in which the transition rates are constant at all ages with* $\mu^{01} = 0.05$, $\mu^{02} = 0.03$, $\mu^{10} = 0.01$ *and* $\mu^{12} = 0.075$. *Assume that the force of interest is* 4% *and*

$$ {}_tp_{40}^{00} = \frac{5e^{-0.06t} + 4e^{-0.105t}}{9} \quad \text{and} \quad {}_tp_{40}^{01} = \frac{10}{9}\left(e^{-0.06t} - e^{-0.105t}\right) \quad \text{for all} \quad t \geq 0. $$

1. *Calculate the actuarial present value of a disability benefit of* 200 *per annum paid continuously to a life now aged* 40 *exact and disabled, during this period of disability, payable to a maximum age of* 60.

2. *The company issues a disability income benefit policy, which pays an* ***inflation–adjusted*** *disability benefit of* $2000e^{0.05t}$ *immediately every time the policyholder becomes disabled at time* t. *Calculate the actuarial present value of the benefit for a healthy life aged* 40.

3. *The policyholder of the policy in (2) pays a level premium of rate* P *continuously while he is healthy, up to age* 50. *Find* P *using the equivalence principle.*

4. *A company also issues a special disability income policy with a benefit of* 360 *per annum payable continuously while the insured is disabled, provided that* ***he has been disabled continuously for at least one year***. *Benefit payments under this policy cease at age* 60.
Calculate the actuarial present value of the benefit for a healthy life aged 40.

Solution:

1. The exit rate for the state of disability is $\nu^1 = \mu^{10} + \mu^{12} = 0.01 + 0.075 = 0.085$. The probability for payment is the occupancy probability ${}_tp_{40}^{\overline{11}} = e^{-\nu^1 t} = e^{-0.085t}$. So the APV is

$$ 200\bar{a}_{40:\overline{20}|}^{\overline{11}} = 200\int_0^{20} e^{-\delta t}\, {}_tp_{40}^{\overline{11}}\, dt = 200\int_0^{20} e^{-0.04t}e^{-0.085t}\, dt = \frac{200}{0.125}\left(1 - e^{-0.125\times 20}\right) = 1468.6640 $$

2. The APV is

$$ \begin{aligned} \int_0^{\infty} e^{-\delta t}\, b_t\, {}_tp_{40}^{00}\, \mu^{01}\, dt &= \int_0^{\infty} e^{-0.04t}\, 2000e^{0.05t}\, \frac{5e^{-0.06t} + 4e^{-0.105t}}{9}\, 0.05\, dt \\ &= \frac{100}{9}\int_0^{\infty} e^{0.01t}\left(5e^{-0.06t} + 4e^{-0.105t}\right) dt \\ &= \frac{500}{9}\int_0^{\infty} e^{-0.05t}\, dt + \frac{400}{9}\int_0^{\infty} e^{-0.095t}\, dt \\ &= \frac{500}{9}\frac{1}{0.05} + \frac{400}{9}\frac{1}{0.095} = \mathbf{1578.947368} \end{aligned} $$

3. The APV of future annuity is

$$ \begin{aligned} \int_0^{10} e^{-\delta t}\, {}_tp_{40}^{00}\, dt &= \int_0^{10} e^{-0.04t}\, \frac{5e^{-0.06t} + 4e^{-0.105t}}{9}\, dt \\ &= \frac{5}{9}\int_0^{10} e^{-0.1t}\, dt + \frac{4}{9}\int_0^{10} e^{-0.145t}\, dt \\ &= \frac{50}{9}\left(1 - e^{-1}\right) + \frac{4}{9}\frac{1}{0.145}\left(1 - e^{-1.45}\right) = \mathbf{5.8579} \end{aligned} $$

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Therefore, by E.P. we get

$$P = \frac{1578.947368}{5.8579} = \mathbf{269.5415367}.$$

4. For benefit to be payable at time t, the insured has to be in state 1 at least 1 year before and continue to stay until t, so the probability is

$$ {}_{t-1}p_{40}^{01}\ p_{40+t-1}^{\overline{11}} = \frac{10}{9}\left(e^{-0.06(t-1)} - e^{-0.105(t-1)}\right) e^{-0.085} $$

So, the APV is

$$\begin{aligned} 360\int_1^{20} e^{-\delta t}\ {}_{t-1}p_{40}^{01}\ p_{40+t-1}^{\overline{11}}\ \mathrm{d}t &= 360\int_1^{20} e^{-0.04t}\ \frac{10}{9}\left(e^{-0.06(t-1)} - e^{-0.105(t-1)}\right) e^{-0.085}\ \mathrm{d}t \\ &= 400e^{-0.04-0.085}\int_1^{20} e^{-0.04(t-1)}\left(e^{-0.06(t-1)} - e^{-0.105(t-1)}\right)\ \mathrm{d}t \\ &= 400e^{-0.125}\int_0^{19} e^{-0.04s}\left(e^{-0.06s} - e^{-0.105s}\right)\ \mathrm{d}s = \mathbf{722.3925}. \end{aligned}$$

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10.7 Calculating Reserves

When we consider the time–t reserve for a policy more general than traditional life insurance, it is clear that different reserves are needed, depending on the state of the insured at time t. These reserves can be calculated using the prospective method.

Example 10.7.1 *Consider the disability income model in which the transition rates are constant and given by* $\mu^{01} = 0.05$, $\mu^{02} = 0.03$, $\mu^{10} = 0.01$ *and* $\mu^{12} = 0.075$. *Assume that the force of interest is* 4% *and*

$$ {}_tp_{40}^{00} = \frac{5e^{-0.06t} + 4e^{-0.105t}}{9} \quad \textit{and} \quad {}_tp_{40}^{01} = \frac{10}{9}\left(e^{-0.06t} - e^{-0.105t}\right) \quad \textit{for all} \quad t \geq 0. $$

The company issues a disability income benefit policy, which pays an ***inflation–adjusted*** *disability benefit of* $2000e^{0.05t}$ *immediately every time the policyholder becomes disabled at time* t. *Calculate the time–*5 *reserve for a life aged* 40 *that is healthy. The policyholder pays a level premium of rate* P *continuously while he is healthy, up to age* 50.

Solution:

Observe first that ${}_tp_{40}^{00} = {}_tp_{45}^{00}$ so the APV of future benefits at time 5 is

$$\begin{aligned} \int_0^{\infty} e^{-\delta t}\ b_{5+t}\ {}_tp_{45}^{00}\ \mu^{01}\ \mathrm{d}t &= \int_0^{\infty} e^{-0.04t}\ 2000e^{0.05(5+t)}\ \frac{5e^{-0.06t} + 4e^{-0.105t}}{9}\ 0.05\ \mathrm{d}t \\ &= \frac{100e^{0.25}}{9}\int_0^{\infty}\left(5e^{-0.05t} + 4e^{-0.095t}\right)\ \mathrm{d}t \\ &= \frac{100e^{0.25}}{9}\left(\frac{5}{0.05} + \frac{4}{0.095}\right) = 2027.4086. \end{aligned}$$

The APV of future annuity is

$$\begin{aligned}\int_0^5 e^{-\delta t}\ {}_tp_{45}^{00}\ \mathrm{d}t &= \int_0^5 e^{-0.04t}\ \frac{5e^{-0.06t}+4e^{-0.105t}}{9}\ \mathrm{d}t\\ &= \frac{5}{9}\int_0^5 e^{-0.1t}\ \mathrm{d}t + \frac{4}{9}\int_0^5 e^{-0.145t}\ \mathrm{d}t\\ &= \frac{50}{9}\left(1-e^{-0.5}\right)+\frac{4}{9}\frac{1}{0.145}\left(1-e^{-0.725}\right)=3.7666.\end{aligned}$$

Therefore, he time–5 reserve is

$${}_5V = 2027.4086 - 269.5415367 \times 3.7666 = 1012.170.$$

Similarly to the case of a single decrement model, when the policy is fully continuous, reserves satisfy a system of ordinary differential equations. This system of Thiele's differential equations, together with the corresponding terminal conditions, can be solved numerically.

For **multiple state models**, we use the following notation:

- $P_t^{(i)}$ stands for annuity–type premium / benefit payable continuously if the life is in state i at time t.
- $b_t^{(ij)}$ stands for benefit payable immediately on a transition from state i to state j at time t.
- ${}_tV^{(i)}$ denotes the benefit reserve at time t, when the state of the insured is i.

For simplicity we ignore expenses in what follows. They can easily be added to the final equation.

System of Thiele's Differential Equation under Multiple State Model

$$\begin{aligned}\frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t} &= P^{(i)} + (\delta_t + \nu_{x+t}^i)\ {}_tV^{(i)} - \sum_{j\neq i}^{n}\left(b_t^{(ij)} +\ {}_tV^{(j)}\right)\ \mu_{x+t}^{ij}\\ &= P^{(i)} + \delta_t\ {}_tV^{(i)} - \sum_{j\neq i}^{n}\left(b_t^{(ij)} +\ {}_tV^{(j)} -\ {}_tV^{(i)}\right)\ \mu_{x+t}^{ij} \text{ (thanks to } \nu_{x+t}^i = \sum_{j\neq i}^{n}\mu_{x+t}^{ij}\text{)}\end{aligned}$$

If the there is a level continuous annuity (could be a survival or a disability) benefit at a rate $S^{(i)}$ the Thiele's Differential Equation becomes

$$\begin{aligned}\frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t} &= P^{(i)} - S^{(i)} + (\delta_t + \nu_{x+t}^i)\ {}_tV^{(i)} - \sum_{j\neq i}^{n}\left(b_t^{(ij)} +\ {}_tV^{(j)}\right)\ \mu_{x+t}^{ij} \qquad (10.3)\\ &= P^{(i)} - S^{(i)} + \delta_t\ {}_tV^{(i)} - \sum_{j\neq i}^{n}\left(b_t^{(ij)} +\ {}_tV^{(j)} -\ {}_tV^{(i)}\right)\ \mu_{x+t}^{ij} \text{ (thanks to } \nu_{x+t}^i = \sum_{j\neq i}^{n}\mu_{x+t}^{ij}\text{)}\end{aligned}$$

Example 10.7.2 *Consider the disability income model and the policy in Example 10.7.1. Write down the system of Thiele's differential equations.*

Solution: We consider shall discuss two cases.

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i. When the state is 0. The exit rate from 0 is $\nu^0 = \mu^{01} + \mu^{02} = 5\% + 3\% = 8\%$, a premium of rate P is received continuously in the first 10 years (and no further premiums afterwards), and the benefits for transition are $b_t^{(01)} = 2000e^{0.05t}$ and $b_t^{(02)} = 0$. Therefore the TDE reads

$$\begin{aligned}
\frac{\mathrm{d}\ {}_tV^{(0)}}{\mathrm{d}t} &= P^{(0)} + (\delta_t + \nu^0_{x+t})\ {}_tV^{(0)} - \sum_{j=1}^{2}\left(b_t^{(0j)} +\ {}_tV^{(j)}\right)\ \mu^{0j}_{x+t} \\
&= P\ \mathbf{1}_{\{t<10\}} + (\delta_t + \nu^0_{x+t})\ {}_tV^{(0)} - \left(2000e^{0.05t} +\ {}_tV^{(1)}\right)\ \mu^{01} -\ {}_tV^{(2)}\ \mu^{02} \\
&= P\ \mathbf{1}_{\{t<10\}} + (0.04 + 0.08)\ {}_tV^{(0)} - \left(2000e^{0.05t} +\ {}_tV^{(1)}\right)\ 0.05 -\ {}_tV^{(2)}\ 0.03 \\
&= P\ \mathbf{1}_{\{t<10\}} + 0.12\ {}_tV^{(0)} - 100e^{0.05t} - 0.05\ {}_tV^{(1)} - 0.03\ {}_tV^{(2)}
\end{aligned}$$

But the reserve for state 2 is ${}_tV^{(2)} = 0$ for any t, finally the TDE becomes

$$\frac{\mathrm{d}\ {}_tV^{(0)}}{\mathrm{d}t} = P\mathbf{1}_{\{t<10\}} - 100e^{0.05t} + 0.12\ {}_tV^{(0)} - 0.05\ {}_tV^{(1)}$$

where $\mathbf{1}_{\{t<10\}}$ stands for the indicator function that is $\mathbf{1}_{\{t<10\}} = 1$ if $t < 10$, and $\mathbf{1}_{\{t<10\}} = 0$ for $t \geq 10$.

ii. When the state is 1. The exit rate is $\nu^1 = \mu^{10} + \mu^{12} = 0.01 + 0.075 = 8.5\%$, no premium is payable, and the benefits for transitions are $b^{(10)} = 0$ and $b^{(12)} = 0$. The TDE is given by

$$\begin{aligned}
\frac{\mathrm{d}\ {}_tV^{(1)}}{\mathrm{d}t} &= P^{(1)} + (\delta_t + \nu^1_{x+t})\ {}_tV^{(1)} - \sum_{j=0,2}\left(b_t^{(1j)} +\ {}_tV^{(j)}\right)\ \mu^{1j}_{x+t} \\
&= (\delta_t + \nu^1_{x+t})\ {}_tV^{(1)} -\ {}_tV^{(0)}\ \mu^{10} -\ 0\ \mu^{12} \\
&= (0.04 + 0.085)\ {}_tV^{(1)} - 0.01\ {}_tV^{(0)} = -0.01\ {}_tV^{(0)} + 0.125\ {}_tV^{(1)}.
\end{aligned}$$

The system above can written as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} {}_tV^{(0)} \\ {}_tV^{(1)} \end{pmatrix} = \begin{pmatrix} P\mathbf{1}_{\{t<10\}} - 100e^{0.05t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0.12 & -0.05 \\ -0.01 & 0.125 \end{pmatrix}\begin{pmatrix} {}_tV^{(0)} \\ {}_tV^{(1)} \end{pmatrix}$$

Set

$${}_t\mathbf{V} = \begin{pmatrix} {}_tV^{(0)} \\ {}_tV^{(1)} \end{pmatrix},\ \mathbf{P}_t = \begin{pmatrix} P\mathbf{1}_{\{t<10\}} - 100e^{0.05t} \\ 0 \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} 0.12 & -0.05 \\ -0.01 & 0.125 \end{pmatrix}$$

the above system becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\ {}_t\mathbf{V} = \mathbf{P}_t + \mathbf{A}\ {}_t\mathbf{V}.$$

and can be solved explicitly or numerically.

For the sake of completeness the system of Thiele's differential equations for the gross premium reserve in a multiple state model is

$$\begin{aligned}
\frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t} &= G_t^{(i)}(1 - c_t^{(i)}) - e_t^{(i)} + (\delta_t + \nu^i_{x+t})\ {}_tV^{(i)} - \sum_{j\neq i}^{n}\left(b_t^{(ij)} + E_t^{(ij)} +\ {}_tV^{(j)}\right)\ \mu^{ij}_{x+t} \\
&= G_t^{(i)}(1 - c_t^{(i)}) - e_t^{(i)} + \delta_t\ {}_tV^{(i)} - \sum_{j\neq i}^{n}\left(b_t^{(ij)} + E_t^{(ij)} +\ {}_tV^{(j)} -\ {}_tV^{(i)}\right)\ \mu^{ij}_{x+t}.
\end{aligned}$$

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10.7.1 Solving Thiele's Differential Equations using Euler's Method

Since we do not know all the initial values for the reserves, the system of Thiele's equations usually cannot be solved by Euler's method by a recursion that starts from 0. For a term policy, the terminal values for all reserves are ${}_nV^{(i)} = 0$ for all i. So, it is possible to perform a backward recursion to calculate reserves using Euler's method for terminal value problem. When adapted to the system of Thiele's equations, the recursion formulas are

$$\frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t} = \frac{{}_tV^{(i)} - {}_{t-h}V^{(i)}}{h} \iff {}_{t-h}V^{(i)} = {}_tV^{(i)} - h\ \frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t}.$$

So for $t = n - kh$ we get

$${}_{n-kh-h}V^{(i)} = {}_{n-kh}V^{(i)} - h\ \left.\frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t}\right|_{t=n-kh}.$$

for different values of i

$$\begin{aligned}
{}_{n-(k+1)h}V^{(0)} &\simeq {}_{n-kh}V^{(0)} - \left.\frac{\mathrm{d}\ {}_tV^{(0)}}{\mathrm{d}t}\right|_{t=n-kh} \times h \\
{}_{n-(k+1)h}V^{(1)} &\simeq {}_{n-kh}V^{(1)} - \left.\frac{\mathrm{d}\ {}_tV^{(1)}}{\mathrm{d}t}\right|_{t=n-kh} \times h \\
&\vdots \\
{}_{n-(k+1)h}V^{(i)} &\simeq {}_{n-kh}V^{(i)} - \left.\frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t}\right|_{t=n-kh} \times h \quad \text{for all } i.
\end{aligned}$$

Now, starting with ${}_nV^{(i)} = 0$ for $i \geq 0$, we can then calculate

$$[{}_{n-h}V^{(0)},\ {}_{n-h}V^{(1)},\ {}_{n-h}V^{(2)}, \ldots], [{}_{n-2h}V^{(0)},\ {}_{n-2h}V^{(1)},\ {}_{n-2h}V^{(2)}, \ldots],\ [{}_{n-3h}V^{(0)},\ {}_{n-3h}V^{(1)},\ {}_{n-3h}V^{(2)}, \ldots]$$

recursively.

Example 10.7.3 *Consider a permanent disability model with three states: State 0: Healthy, State 1: Permanently disabled, and State 2: Dead. Suppose that* $\mu_x^{01} = 0.02$, $\mu_x^{02} = 0.04$, $\mu_x^{12} = 0.05$ *for* $x \geq 40$. *An insurance company uses the model to calculate premiums for a 6–year term insurance policy issued to a life aged 40 who is now healthy. The death benefit is* $100,000$, *payable at the moment of death, with a disability income payable continuously at a rate of* $5,000$ *if the life is permanently disabled. Premiums are payable continuously provided that the policyholder is healthy.*

1. *Calculate the level premium rate for this policy on the following basis:*
   a. *Interest: constant 5% force of interest*
   b. *Expenses: nil*
   c. *Survival model: Transition intensities as given above.*

2. *Let* ${}_tV^{(0)}$ *and* ${}_tV^{(1)}$ *be the benefit reserve at time t, when the policyholder is in state 0 and state 1, respectively. Calculate* ${}_3V^{(0)}$ *and* ${}_3V^{(1)}$. *Assume that the premium basis and reserve basis are the same.*

3. *Write down the system of Thiele's differential equation that is satisfied by* ${}_tV^{(0)}$ *and* ${}_tV^{(1)}$.

Solution:

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1. $\mathbf{APV}(\mathbf{F.P.})_0 = P\bar{a}^{00}_{40:\overline{6}|}$ and $\mathbf{APV}(\mathbf{F.B.})_0 = 5000\bar{a}^{01}_{40:\overline{6}|} + 100000\bar{A}^{02}_{40:\overline{6}|}$. Remember first that

$$ {}_tp^{00}_{40} = e^{-(0.02+0.04)t} = e^{-0.06t} $$

thus

$$ \bar{a}^{00}_{40:\overline{6}|} = \int_0^6 e^{-\delta t}\ {}_tp^{00}_{40}\ \mathrm{d}t = \int_0^6 e^{-0.05t}\ e^{-0.06t}\ \mathrm{d}t = \int_0^6 e^{-0.11t}\ \mathrm{d}t = \frac{1-e^{-0.66}}{0.11} = 4.3923. $$

Moreover

$$ \begin{aligned} {}_tp^{01}_{40} &= \int_0^t {}_sp^{00}_{40}\ \mu^{01}\ {}_{t-s}p^{11}_{40+s}\ \mathrm{d}s = 0.02\int_0^t e^{-0.06s}e^{-0.01(t-s)}\ \mathrm{d}s \\ &= 0.02e^{-0.05t}\int_0^t e^{-0.01s}\ \mathrm{d}s = 2e^{-0.05t}\left(1-e^{-0.01t}\right), \end{aligned} $$

$$ \begin{aligned} \bar{a}^{01}_{40:\overline{6}|} &= \int_0^6 e^{-\delta t}\ {}_tp^{01}_{40}\ \mathrm{d}t = \int_0^6 e^{-0.05t}\ 2e^{-0.05t}\left(1-e^{-0.01t}\right)\ \mathrm{d}t \\ &= 2\int_0^6 e^{-0.1t}\left(1-e^{-0.01t}\right)\ \mathrm{d}t = 0.23925. \end{aligned} $$

And

$$ \begin{aligned} \bar{A}^{02}_{40:\overline{6}|} &= \int_0^6 e^{-\delta t}\sum_{k=0}^{1}{}_tp^{0k}_{40}\ \mu^{k2}_{40+t}\ \mathrm{d}t = \int_0^6 e^{-\delta t}\ {}_tp^{00}_{x}\ \mu^{02}_{40+t}\ \mathrm{d}t + \int_0^6 e^{-\delta t}\ {}_tp^{01}_{x}\ \mu^{12}_{x+t}\ \mathrm{d}t \\ &= \int_0^6 e^{-0.05t}\ e^{-0.06t}\ 0.04\ \mathrm{d}t + \int_0^6 e^{-0.05t}\ \left(2e^{-0.05t}\left(1-e^{-0.01t}\right)\right)0.05\ \mathrm{d}t \\ &= 0.04\int_0^6 e^{-0.11t}\ \mathrm{d}t + 0.1\int_0^6 e^{-0.1t}\ \mathrm{d}t - 0.1\int_0^6 e^{-0.11t}\ \mathrm{d}t = 0.187652 \end{aligned} $$

Finally

$$ P = \frac{5000\bar{a}^{01}_{40:\overline{6}|} + 100000\bar{A}^{02}_{40:\overline{6}|}}{\bar{a}^{00}_{40:\overline{6}|}} = \frac{5000\times 0.23925 + 18765.2}{4.3923} = 4544.6. $$

2. (b) (i)

$$ \begin{aligned} \bar{a}^{00}_{43:\overline{3}|} &= \frac{1-e^{-0.33}}{0.11} = 2.5552388, \\ \bar{A}^{02}_{43:\overline{3}|} &= \frac{5}{11} - e^{-0.3} + \frac{6}{11}e^{-0.33} = 0.10586745 \\ \bar{a}^{01}_{43:\overline{3}|} &= \frac{2}{0.1}\left(1-e^{-0.3}\right) - \frac{2}{0.11}\left(1-e^{-0.33}\right) = 0.0731580 \end{aligned} $$

The reserve is

$$ {}_3V^{(0)} = 100000(0.10586745) + 5000(0.0731580) - 4544.699(2.5552388) \ = \ -660.256 $$

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(ii) $\bar{a}^{11}_{43:\overline{3}|} = \int_0^3 e^{-0.05t}\ {}_tp^{11}_{43}\ \mathrm{d}t = \int_0^3 e^{-0.05t}e^{-0.05t}\ \mathrm{d}t = \frac{1-e^{-0.3}}{0.1} = 2.5918178$

$$\bar{A}^{12}_{43:\overline{3}|} = \int_0^3 e^{-0.05t}\ 0.05e^{-0.05t}\ \mathrm{d}t = \frac{0.05}{0.10}\left(1-e^{-0.3}\right) = 0.129591.$$

The reserve is

$${}_3V^{(1)} = 100000(0.129591) + 5000(2.5901878) = 25910.04$$

3. Notice that $S^{(0)} = 0$ and $S^{(1)} = 5000$, hence from the general formula (10.3) we have

$$\begin{aligned}
\frac{\mathrm{d}\ {}_tV^{(i)}}{\mathrm{d}t} &= P^{(i)} - S^{(i)} + \left(\delta_t + \nu^i_{x+t}\right)\ {}_tV^{(i)} - \sum_{j=0,j\neq i}^{n}\left(b_t^{(ij)} +\ {}_tV^{(j)}\right)\mu^{ij}_{x+t}\\
\frac{\mathrm{d}\ {}_tV^{(0)}}{\mathrm{d}t} &= P^{(0)} - 0 + \left(\delta_t + \nu^0_{x+t}\right)\ {}_tV^{(0)} - \sum_{j=1}^{2}\left(b_t^{(0j)} +\ {}_tV^{(j)}\right)\mu^{0j}_{x+t}\\
&= P^{(0)} + \left(\delta_t + \nu^0_{x+t}\right)\ {}_tV^{(0)} - \left(b_t^{(01)} +\ {}_tV^{(1)}\right)\mu^{01}_{x+t} - \left(b_t^{(02)} +\ {}_tV^{(2)}\right)\mu^{02}_{x+t}\\
&= 4544.699 + (0.05+0.06)\ {}_tV^{(0)} - \left(0 +\ {}_tV^{(1)}\right)0.02 - (100000+0)\,(0.04)\\
&= 544.70 + 0.11\ {}_tV^{(0)} - 0.02\ {}_tV^{(1)}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\ {}_tV^{(1)}}{\mathrm{d}t} &= P^{(1)} - 5000 + \left(\delta_t + \nu^1_{x+t}\right)\ {}_tV^{(1)} - \left(b_t^{(10)} +\ {}_tV^{(0)}\right)\mu^{10}_{x+t} - \left(b_t^{(12)} +\ {}_tV^{(2)}\right)\mu^{12}_{x+t}\\
&= 0 - 5000 + (0.05+0.05)\ {}_tV^{(1)} - \left(0 +\ {}_tV^{(0)}\right)0 - (100000+0)\,0.05\\
&= -10000 + 0.1\ {}_tV^{(1)},
\end{aligned}$$

Notice that we have used the fact that ${}_tV^{(2)} = 0$ (since after death the contract is terminated or settled). Moreover you can justify the result using the prospective approach.

Example 10.7.4 *Consider a permanent disability model with three states: State 0: Healthy, State 1: Permanently disabled, and State 2: Dead. Suppose that* $\mu^{01}_x = 0.02$, $\mu^{02}_x = 0.04$, $\mu^{12}_x = 0.05$ *for* $x \geq 40$. *An insurance company uses the model to calculate premiums for a 6-year term insurance policy issued to a life aged 40 who is now healthy. The death benefit is 100,000, payable at the moment of death. Premiums are payable monthly in advance provided that the policyholder is healthy.*

1. *Calculate the monthly premium for this policy on the following basis:*
   - *(a) Interest: constant force of interest of 6% per year*
   - *(b) Initial expense: 60% of the gross premium*
   - *(c) Renewal expenses: 10% of each of the premium except the first*
   - *(d) Survival model: Transition intensities as given above.*

2. *Calculate the gross premium reserve at time 3 for a policyholder who is permanently disabled by that time. The reserve basis is the same as the premium basis, with the exception that:*
   (i) $\mu^{12}_x = 0.06$ *for* $x \geq 40$;
   (ii) the constant force of interest is 5% per year;
   (iii) the initial expense is 100% of the gross premium.

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Solution:
(a) The transition probabilities have been calculated in the previous question. Let the monthly premium by G use the notation $\ddot{a}^{00(12)}_{40:\overline{6}|}$ for the APV of a monthly life annuity of $\frac{1}{12}$ and $\bar{A}^{ik}_{40:\overline{6}|}$ for the APV of continuous multiple state term life insurance policies whenever transition occurs from i to k. Now, by the equivalence principle, we have

$$12G\ddot{a}^{00(12)}_{40:\overline{6}|} = 12\times 0.1G\left(\ddot{a}^{00(12)}_{40:\overline{6}|} - \frac{1}{12}\right) + 0.6G + 100000\bar{A}^{02}_{40:\overline{6}|}$$

that is

$$12\times 0.9G\ddot{a}^{00(12)}_{40:\overline{6}|} + 0.1G - 0.6G = 100000\bar{A}^{02}_{40:\overline{6}|} \tag{10.4}$$

Now, let us calculate $\bar{A}^{02}_{40:\overline{6}|}$ and $\ddot{a}^{00(12)}_{40:\overline{6}|}$

$$\begin{aligned}
\bar{A}^{02}_{40:\overline{6}|} &= \int_0^6 e^{-0.06t}\left({}_tp^{00}_{40}\mu^{02}_{40+t} + {}_tp^{01}_{40}\mu^{12}_{40+t}\right)\mathrm{d}t\\
&= \int_0^6 e^{-0.06t}\left(0.04e^{-0.06t} + 2\left(e^{-0.05t}-e^{-0.06t}\right)0.05\right)\mathrm{d}t\\
&= \int_0^6 e^{-0.06t}\left(0.1e^{-0.05t} - 0.06e^{-0.06t}\right)\mathrm{d}t\\
&= \frac{0.1}{0.11}\left(1-e^{-0.66}\right) - \frac{0.06}{0.12}\left(1-e^{-0.72}\right) = 0.1826022
\end{aligned}$$

and

$$\begin{aligned}
\ddot{a}^{00(12)}_{40:\overline{6}|} &= \frac{1}{12}\sum_{k=0}^{12\times 6-1} v^{\frac{k}{12}}\,{}_{\frac{k}{12}}p^{00}_{40} = \frac{1}{12}\sum_{k=0}^{71} e^{-0.06\frac{k}{12}}\,e^{-(0.02+0.04)\frac{k}{12}}\\
&= \frac{1}{12}\sum_{k=0}^{71} e^{-0.01k} = \frac{1}{12}\frac{1-e^{-0.01\times 72}}{1-e^{-0.01}} = 4.2985.
\end{aligned}$$

substituting n (10.4) we end up with $G = \mathbf{397.6213}$.
(b) Since the life is already disabled and Y can never revert back to state 0, there is no future premium. The gross premium reserve is equal to the sum of the APVs of the death benefit at time 3:

$${}_3V^{(1)} = 100000\,\bar{A}^{12}_{43:\overline{3}|}$$

where

$$\bar{A}^{12}_{43:\overline{3}|} = \int_0^3 e^{-0.05t}\,{}_tp^{11}_{40}\,\mu^{12}_{40+t}\,\mathrm{d}t = \int_0^3 e^{-0.05t}\,e^{-0.06t}\,0.06\,\mathrm{d}t = \frac{0.06}{0.11}(1-e^{-0.33}) = 0.1533143.$$

So, the gross premium reserve is ${}_3V^{(1)} = 100000\times 0.1533143 = \mathbf{15331.43}$.
As an easy exercise calculate ${}_3V^{(0)}$.

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