Math 244 - Linear Algebra

Chapter 7: Diagonalizable Matrices

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Definition

If A is an $n \times n$ matrix, then a nonzero vector X in $\mathbb{R}^n$ is called an **eigenvector** of A (or of the matrix operator T_A) if Ax is a scalar multiple of x; that is, $Ax = \lambda x$ for some scalar λ. The scalar λ is called an **eigenvalue** of A (or of T_A), and x is said to be an **eigenvector corresponding** to λ.

Examples:

1. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ corresponding to 3.
2. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ corresponding to 0.
3. $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ corresponding to 2.

How to find Eigenvalues of a matrix

Theorem

If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation $\det(A - \lambda I) = 0$. This is called the characteristic equation of A.

Examples:

1. For $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, we have $\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda)$. the characteristic equation is then $(\lambda + 1)(\lambda - 3) = 0$ and the eigenvalues are $\lambda = -1, 3$.
2. For $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, we have $\det(A - \lambda I) = (1 - \lambda)^2 - 1 = \lambda(\lambda - 2)$. the characteristic equation is then $\lambda(\lambda - 2) = 0$ and the eigenvalues are $\lambda = 0, 2$.

How to find Eigenvectors of a matrix

Remark

If λ is an eigenvalue of a square matrix A, to find the corresponding eigenvectors, we solve the homogeneous linear system given by the matricial equation $(A - \lambda I)X = 0$.

Examples: For the matrix $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$,

1. For the eigenvalue $\lambda = -1$, we solve the system: $\begin{cases} 4x & = 0 \\ 8x & = 0 \end{cases}$, which is equivalent to $x = 0$. The eigenvectors are $X = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.
2. For the eigenvalue $\lambda = 3$, we solve the system: $\begin{cases} 0 & = 0 \\ 8x - 4y & = 0 \end{cases}$, which is equivalent to $y = 2x$. The eigenvectors are $X = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$.

The Eigenspace of a matrix corresponding to a given eigenvalue

Theorem

Let λ be an eigenvalue of an $n \times n$ matrix A. The set

$$E_\lambda = \{X \in \mathcal{M}_{n,1}(\mathbb{R}) | AX = \lambda X\}$$

of the corresponding eigenvectors, is a subspace of $\mathcal{M}_{n,1}(\mathbb{R})$ and satisfies $1 \leq \dim E_\lambda \leq m_\lambda$, where m_λ is the multiplicity of λ in the characteristic equation of A.
E_λ is called the eigenspace *of the matrix A corresponding to the eigenvalue λ, m_λ the* algebraic multiplicity *of λ and $\dim E_\lambda$ its* geometric multiplicity.

The Eigenspace of a matrix corresponding to a given eigenvalue

Examples:

1. For $A = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$, the characteristic equation is $(3-\lambda)^2 = 0$. It has a unique eigenvalue $\lambda = 3$ with algebraic multiplicity $m_3 = 2$.
   The corresponding eigenspace is the solution set of the equation $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and therefore $E_3 = \left\{ t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$. The geometric multiplicity of 3 is $\dim E_3 = 1$.

The Eigenspace of a matrix corresponding to a given eigenvalue

2. For $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, the characteristic equation is $(3-\lambda)^2(1-\lambda) = 0$. It has two eigenvalues: $\lambda = 3, 1$ with algebraic multiplicities $m_3 = 2$ and $m_1 = 1$.
   The eigenspace corresponding to $\lambda = 3$ is the solution set of the equation $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and therefore $E_3 = \left\{ s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$. The geometric multiplicity of 3 is $\dim E_3 = 2$.

Diagonalizable matrices

Theorem

Let $X_1, X_2, \ldots, X_k$ be eigenvectors corresponding to distinct eigenvalues of a square matrix A. The family $\{X_1, X_2, \ldots, X_k\}$ is linearly independent.

Definition

We say that an $n \times n$ matrix A is a diagonalizable if the space $\mathcal{M}_{n,1}$ has a basis of eigenvectors of A.

Theorem

If an $n \times n$ matrix has n different eigenvalues, it is diagonalizable.

Diagonalizable matrices

Theorem

Let A be an $n \times n$ matrix. The following statements are equivalent.

1. *A is diagonalizable.*
2. *The characteristic equation of A has n zeros counted with their multiplicities and each eigenenvalue of A has equal geometric and algebraic multiplicies.*
3. *There exist an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.*

Proof: (1) $\Leftrightarrow$ (3) If C is the standard basis of $\mathcal{M}_{n,1}(\mathbb{R})$ and B a basis of eigenvectors of A, then take $P = P_{B \leftarrow C}$. The matrix $P^{-1}AP$ is diagonal with entries the eigenvalues of A corresponding to the eigenvectors in B. For the converse, the columns of P are eigenvectors of A and they form a basis of $\mathcal{M}_{n,1}(\mathbb{R})$.

Proof: (2) $\Rightarrow$ (1) Consider for each eigenspace E_λ a basis. Put all these bases of eigenspaces together to form a basis of eigenvectors of $\mathcal{M}_{n,1}(\mathbb{R})$ since

$$\dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_k} = m_{\lambda_1} + m_{\lambda_2} + \cdots + m_{\lambda_k} = n.$$

Remark

If $A = PDP^{-1}$, then $A^m = PD^mP^{-1}$ for any $m \in \mathbb{Z}$. In particular, any power of a diagonalizable matrix is diagonalizable.

Theorem

Any symmetric matrix is diagonalizable and has an orthonormal basis of eigenvectors.

The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is skew-symmetric and not diagonalizable since its characteristic equation is $\lambda^2 + 1 = 0$ which has no real zeros.