Math 244 - Linear Algebra

Chapter 6: General Linear Transformations

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November 19, 2025

Definition of a General Linear Transformation

Definition

If $T : V \longrightarrow W$ is a mapping from a vector space V to a vector space W, then T is called a **linear transformation** from V to W if the following two properties hold for all vectors $\vec{u}$ and $\vec{v}$ in V and for all scalars k:

1. $T(k\vec{u}) = kT(\vec{u})$ **[Homogeneity property]**
2. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ **[Additivity property]**

In the special case where $V = W$, the linear transformation T is called a **linear operator** on the vector space V.

Examples of General Linear Transformations:

- Given a matrix A of size $m \times n$, the mapping $T_A : \mathcal{M}_{n,1} \longrightarrow \mathcal{M}_{m,1}$ defined by $T_A(X) = AX$, for $X \in \mathcal{M}_{n,1}$ is a linear transformation.
- The mapping $T : \mathcal{P}_3 \longrightarrow \mathbb{R}^4$ defined by $T(P) = (P(-1), P(0), P(1))$ is a linear transformation.
- The identity mapping $Id : V \longrightarrow V$ defined by $Id(\vec{v}) = \vec{v}$, for $\vec{v} \in V$, is a linear transformation.

Remark

If $T : V \longrightarrow W$ is a linear transformation, then:

$$T(k_1\vec{u}_1 + k_2\vec{u}_2) = k_1 T(\vec{u}_1) + k_2 T(\vec{u}_2)$$

for all $\vec{u}_1$ and $\vec{u}_2$ in V and k_1, k_2 scalars.
More general:

$$T(k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_n\vec{u}_n) = k_1 T(\vec{u}_1) + k_2 T(\vec{u}_2) + \cdots + k_n T(\vec{u}_n)$$

for all $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in V and $k_1, k_2, \ldots, k_n$ scalars.

Theorem

If $T : V \longrightarrow W$ is a linear transformation, then:

1. $T(\vec{0}) = \vec{0}$.
2. $T(\vec{u}_1 - \vec{u}_2) = T(\vec{u}_1) - T(\vec{u}_2)$ *for all $\vec{u}_1$ and $\vec{u}_2$ in V.*

Kernel and Range of a Linear Transformation

Definition

If $T : V \longrightarrow W$ is a linear transformation, then the set of vectors in V that T maps into $\vec{0}$ is called the kernel of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the range of T and is denoted by $R(T)$.

Theorem

If $T : V \longrightarrow W$ is a linear transformation, then:

1. *The kernel of T is a subspace of V.*
2. *The range of T is a subspace of W.*

Rank and nullity of a Linear Transformation

Definition

Let $T : V \longrightarrow W$ be a linear transformation. If the range of T is finite-dimensional, then its dimension is called the **rank** of T; and if the kernel of T is finite-dimensional, then its dimension is called the **nullity** of T. The rank of T is denoted by $rank(T)$ and the nullity of T by $nullity(T)$.

Theorem (Dimension Theorem for Linear Transformations)

If $T : V \longrightarrow W$ is a linear transformation from a finite-dimensional vector space V to a vector space W, then the range of T is finite-dimensional, and

$$rank(T) + nullity(T) = \dim V.$$

One-to-one Transformations

Definition

If $T : V \longrightarrow W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **one-to-one** if T maps distinct vectors in V into distinct vectors in W.

Theorem

If $T : V \longrightarrow W$ is a linear transformation, then the following statements are equivalent.

1. *T is one-to-one.*
2. *$ker(T) = \{\vec{0}\}$.*
3. *If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subset V$ is linearly independent, then $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subset W$ is linearly independent.*

Onto Transformations

Definition

If $T : V \longrightarrow W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **onto** (or onto W) if every vector in W is the image of at least one vector in V.

Theorem

If $T : V \longrightarrow W$ is a linear transformation, then the following statements are equivalent.

1. *T is onto.*
2. *If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is spanning V, then $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is spanning W.*

One-to-one and Onto Transformations

Theorem

Let V and W be finite-dimensional vector spaces, and $T : V \longrightarrow W$ a linear transformation.

1. *If T is one-to-one, then* $\dim V \leq \dim W$.
2. *If T is onto, then* $\dim V \geq \dim W$.

Theorem

If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \longrightarrow W$ is a linear transformation, then the following statements are equivalent.

1. *T is one-to-one.*
2. *T is onto.*

Definition

If $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$ are linear transformations, then the composition of T_2 with T_1, denoted by $T_2 \circ T_1$ (which is read " T_2 circle T_1"), is the function defined by the formula

$$T_2 \circ T_1(\vec{u}) = T_2(T_1(\vec{u}))$$

where $\vec{u}$ is a vector in U.

Theorem

If $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$ are linear transformations, then $T_2 \circ T_1 : U \longrightarrow W$ is also a linear transformation.

Remark: If A and B are two matrices of size $m \times n$ and $n \times p$, respectively, then $T_A \circ T_B = T_{AB}$.

Inverse transformation and its properties

Definition

An isomorphism is a one-to-one and onto linear transformation.

Definition

Let $T : U \longrightarrow V$ be a linear transformation. We say that T is invertible if there exists a mapping $S : V \longrightarrow U$, called an iverse of T, such that $S \circ T = Id_U$ and $T \circ S = Id_V$.

Theorem

A linear transformation is invertible if and only if it is an isomorphism. In this case, the inverse of T is unique, denoted T^{-1} and is an isomorphism.

Inverse transformation and its properties

Theorem

Let $T : U \longrightarrow V$ be an isomorphism of finite dimensional spaces U and V. We have:

1. $\dim V = \dim W$ *and* $(T^{-1})^{-1} = T$.
2. *If T_1, T_2 are invertible, then $T_2 \circ T_1$ is invertible and we have* $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

Theorem

Let $T : V \longrightarrow W$ be a linear transformation, B a basis of V and C a basis of W. There exists a matrix denoted $[T]_{C,B}$ satisfying for any $\vec{v} \in V$

$$[T(\vec{v})]_C = [T]_{C,B}[\vec{v}]_B.$$

If $V = W$ and $B = C$, we write $[T]_B$.

Example: For $T : \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(P(X)) = P'(X) + P(0)X^2$ and $B = C = (1, X, X^2)$, we have for $P(X) = aX^2 + bX + c$,

$$[T(P(X))]_C = [P'(X) + P(0)X^2]_C = [2aX + b + cX^2]_C = \begin{pmatrix} b \\ 2a \\ c \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = [T]_{C,B}[P(X)]_B.$$

Theorem

Let $T: V \longrightarrow W$ be a linear transformation, $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ a basis of V and C a basis of W. The matrix $[T]_{C,B}$ associated to the linear transformation T with respect to the bases B and C is given by:

$$[T]_{C,B} = [[T(\vec{v}_1)]_C | T([\vec{v}_2)]_C | \cdots | [T(\vec{v}_n)]_C].$$

Examples: For $T: \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ defined by $T(P(X)) = P'(X)$, for all $P(X) \in \mathcal{P}_2$, and $B = C = (1, X, X^2)$, we have $(1)' = 0$, $X' = 1$, $(X^2)' = 2X$ and $[T]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Theorem

If $T_1 : U \longrightarrow V$ *and* $T_2 : V \longrightarrow W$ *are linear transformations, and if* B_1, B_2, *and* B_3 *are bases for* U, V, *and* W, *respectively , then* $[T_2 \circ T_1]_{B_3,B_1} = [T_2]_{B_3,B_2}[T_1]_{B_2,B_1}$.

Theorem

Let $T : V \longrightarrow W$ *be a linear transformation,* B *a basis of* V *and* C *a basis of* W.

1. T *is one-to-one if and only if the matrix* $[T]_{C,B}$ *has a left inverse.*
2. T *is onto if and only if the matrix* $[T]_{C,B}$ *has a right inverse.*
3. T *is an isomorphism if and only if the matrix* $[T]_{C,B}$ *is invertible. In this case* $[T^{-1}]_{B,C} = [T]_{C,B}^{-1}$.

Example: Let $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ be defined by $T(P(X)) = XP'(X) + P(1)$ for $P(X) \in \mathcal{P}_2$. T is an isomorphism and we have $T^{-1}(P(X)) = \int_1^X \frac{P(t)-P(0)}{t}dt + P(0)$ for $P(X) \in \mathcal{P}_2$.
Consider $B = (1, X, X^2)$.
Because $T(1) = 1$, $T(X) = X + 1$ and $T(X^2) = 2X^2 + 1$, we find

$$[T]_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Because $T^{-1}(1) = 1$, $T^{-1}(X) = X - 1$ and $T^{-1}(X^2) = \frac{1}{2}X^2 - \frac{1}{2}$,

we find $[T^{-1}]_B = \begin{pmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$.

Clearly $[T]_B[T^{-1}]_B = I$. Thus $[T^{-1}]_B = [T]_B^{-1}$.

Theorem

Let $T : V \longrightarrow W$ be a linear transformation, B and B' two bases of V and C and C' two bases of W. We have

$$[T]_{C',B'} = P_{C' \leftarrow C}[T]_{C,B}P_{B \leftarrow B'}.$$

Example: Let $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ be defined for $P(X) \in \mathcal{P}_2$ by $T(P(X)) = XP'(X) + P(1)$. Let $B = (X^2 - 1, X^2 - X, X^2 + X)$ and $B' = (1, X, X^2)$.

Because $T(X^2 - 1) = 2X^2 = (X^2 - X) + (X^2 + X)$,
$T(X^2 - X) = 2X^2 - X = \frac{3}{2}(X^2 - X) + \frac{1}{2}(X^2 + X)$ and
$T(X^2 + X) = 2X^2 + X + 2 = -2(X^2 - 1) + \frac{3}{2}(X^2 - X) + \frac{5}{2}(X^2 + X)$,
we find $[T]_B = \begin{pmatrix} 0 & 0 & -2 \\ 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & \frac{5}{2} \end{pmatrix}$. Moreover $[T]_{B'} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

$P_{B' \leftarrow B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $P_{B \leftarrow B'} = P_{B' \leftarrow B}^{-1} = \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

We check and find that $[T]_{B'} = P_{B' \leftarrow B}[T]_B P_{B \leftarrow B'}$.