Math 244 - Linear Algebra

Chapter 5: Inner Product

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Definition of an inner product and examples

Definition

An **inner product** on a real vector space V is a function that associates a real number $< \vec{u}, \vec{v} >$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars k.

1. $< \vec{u}, \vec{v} >=< \vec{v}, \vec{u} >$ [Symmetry axiom]
2. $< \vec{u} + \vec{v}, \vec{w} >=< \vec{u}, \vec{w} > + < \vec{v}, \vec{w} >$ [Additivity axiom]
3. $< k\vec{u}, \vec{v} >= k < \vec{u}, \vec{v} >$ [Homogeneity axiom]
4. $\vec{u}^2 =< \vec{u}, \vec{u} >\geq 0$ and $< \vec{u}, \vec{u} >= 0$ if and only if $\vec{u} = 0$ [Positivity axiom]

A real vector space with an inner product is called a **real inner product space**.

Examples of real inner products:

- On $\mathbb{R}^3$, $< (x_1, y_1, z_1), (x_2, y_2, z_2) >= x_1x_2 + y_1y_2 + z_1z_2$. This is called the standard inner product of $\mathbb{R}^3$.
- On $\mathbb{R}^3$, $< (x_1, y_1, z_1), (x_2, y_2, z_2) >= 2x_1x_2 + 5y_1y_2 + 3z_1z_2$. In general $< (x_1, y_1, z_1), (x_2, y_2, z_2) >= ax_1x_2 + by_1y_2 + cz_1z_2$ with $a, b, c > 0$ is an inner product. On $\mathbb{R}^3$, $< (x_1, y_1, z_1), (x_2, y_2, z_2) >= x_1x_2 + ay_1y_2 + 3z_1z_2 + x_1y_2 + y_1x_2$ is an inner product iff $a > 1$.
- On $\mathcal{P}_2$, $< P_1(X), P_2(X) >= a_1a_2 + b_1b_2 + c_1c_2$ is the standard inner product where $P_i(X) = a_i + b_iX + c_iX^2, i = 1, 2$.
- On $\mathcal{P}_2$, $< P(X), Q(X) >= P(0)Q(0) + P(1)Q(1) + P(2)Q(2)$ is called the evaluation inner product on $\mathcal{P}_2$.
- On $\mathcal{M}_2(\mathbb{R})$, $< A, B >= tr(A^tB)$.

Definition

If V is a real inner product space, then:

- The **norm** (or length) of a vector $\vec{v}$ in V is denoted by $\|\vec{v}\|$ and is defined by $\|\vec{v}\| = \sqrt{\vec{v}^2}$.
- The **distance** between two vectors is denoted by $d(u, v)$ and is defined by $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(\vec{u} - \vec{v})^2}$.

A vector of norm 1 is called a **unit vector**.

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in a real inner product space V, and if k is a scalar, then:

1. $\|\vec{v}\| \geq 0$ *with equality if and only if* $\vec{v} = \vec{0}$.
2. $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$.
3. $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$.
4. $d(\vec{u}, \vec{v}) \geq 0$ *with equality if and only if* $\vec{u} = \vec{v}$.

Theorem (Cauchy–Schwarz Inequality)

If $\vec{u}$ and $\vec{v}$ are vectors in a real inner product space V, then

$$| < \vec{u}, \vec{v} > | \leq \|\vec{u}\| \cdot \|\vec{v}\|,$$

and equality holds if and only if the vectors $\vec{u}$ and $\vec{v}$ are proportional.

This enables us to define the angle θ between two vectors $\vec{u}$ and $\vec{v}$ to be

$$\theta = \cos^{-1}\left(\frac{< \vec{u}, \vec{v} >}{\|\vec{u}\| \cdot \|\vec{v}\|}\right).$$

Definition

Two vectors $\vec{u}$ and $\vec{v}$ in an inner product space V are called orthogonal if $< \vec{u}, \vec{v} >= 0$.

Theorem

If $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors in a real inner product space V, and if k is any scalar, then:

1. $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ *[Triangle inequality for vectors]*
2. $d(\vec{u},\vec{v}) \leq d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$ *[Triangle inequality for distances]*
3. $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$ *[The parallelogram identity]*

Theorem (Generalized Theorem of Pythagoras)

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal in a real inner product space if and only if

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Definition

A set of two or more vectors in a real inner product space is said to be orthogonal if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be orthonormal.

Theorem

If S is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Orthogonal and orthonormal bases

Theorem

1. *If* $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ *is an orthogonal basis for an inner product space* V*, and if* $\vec{u}$ *is any vector in* V*, then*

$$\vec{u} = \frac{< \vec{u}, \vec{v}_1 >}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{< \vec{u}, \vec{v}_2 >}{\|\vec{v}_2\|^2}\vec{v}_2 + \cdots + \frac{< \vec{u}, \vec{v}_n >}{\|\vec{v}_n\|^2}\vec{v}_n,$$

$$\text{and } \|\vec{u}\|^2 = \frac{< \vec{u}, \vec{v}_1 >^2}{\|\vec{v}_1\|^2} + \frac{< \vec{u}, \vec{v}_2 >^2}{\|\vec{v}_2\|^2} + \cdots + \frac{< \vec{u}, \vec{v}_n >^2}{\|\vec{v}_n\|^2}.$$

2. *If* $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ *is an orthonormal basis for an inner product space* V*, and if* $\vec{u}$ *is any vector in* V*, then*

$$\vec{u} =< \vec{u}, \vec{v}_1 > \vec{v}_1 + < \vec{u}, \vec{v}_2 > \vec{v}_2 + \cdots + < \vec{u}, \vec{v}_n > \vec{v}_n,$$

$$\text{and } \|\vec{u}\|^2 =< \vec{u}, \vec{v}_1 >^2 + < \vec{u}, \vec{v}_2 >^2 + \cdots + < \vec{u}, \vec{v}_n >^2 .$$

Theorem

Every nonzero finite-dimensional inner product space has an orthonormal basis.

Theorem (The Gram–Schmidt Process)

To convert a basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ *into an orthogonal basis* $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ *, perform the following computations:*

Step 1. $\vec{v}_1 = \vec{u}_1,$

Step 2. $\vec{v}_2 = \vec{u}_2 - \frac{<\vec{u}_2,\vec{v}_1>}{\|\vec{v}_1\|^2}\vec{v}_1,$

Step 3. $\vec{v}_3 = \vec{u}_3 - \frac{<\vec{u}_3,\vec{v}_1>}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{<\vec{u}_3,\vec{v}_2>}{\|\vec{v}_2\|^2}\vec{v}_2,$

Step 4. $\vec{v}_4 = \vec{u}_4 - \frac{<\vec{u}_4,\vec{v}_1>}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{<\vec{u}_4,\vec{v}_3>}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{<\vec{u}_4,\vec{v}_3>}{\|\vec{v}_3\|^2}\vec{v}_3,$

$\vdots$

(continue for r steps)

To convert the orthogonal basis into an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_r\}$, normalize the orthogonal basis vectors $\vec{q}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$.