Math 244 - Linear Algebra

Chapter 4: Vector Spaces

Dr. Malik Talbi
King Saud University, Mathematic's Department

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Definition of a real vector space and examples

A **real vector space** $(V, +, \cdot)$ is a nonempty set V together with two operations $+$, called **addition**, and $\cdot$, called **multiplication by a scalar**, satisfying the following **axioms**:

1. $\forall \vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} \in V$;
2. $\forall \vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} = \vec{v} + \vec{u}$;
3. $\forall \vec{u}, \vec{v}, \vec{w} \in V$, we have $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$;
4. There exists an element $\vec{0}$ in V, called a **zero vector**, such that $\forall \vec{u} \in V$, we have $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$;
5. $\forall \vec{u} \in V$, there exists an element $-\vec{u} \in V$, called a **negative** of $\vec{u}$, such that $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$;
6. $\forall k \in \mathbb{R}$, $\forall \vec{u} \in V$, we have $k \cdot \vec{u} \in V$;
7. $\forall k \in \mathbb{R}$, $\forall \vec{u}, \vec{v} \in V$, we have $k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$;
8. $\forall k_1, k_2 \in \mathbb{R}$, $\forall \vec{u} \in V$, we have $(k_1 + k_2) \cdot \vec{u} = k_1 \cdot \vec{u} + k_2 \cdot \vec{u}$;
9. $\forall k_1, k_2 \in \mathbb{R}$, $\forall \vec{u} \in V$, we have $(k_1 k_2) \cdot \vec{u} = k_1 \cdot (k_2 \cdot \vec{u})$;
10. $\forall \vec{u} \in V$, we have $1 \cdot \vec{u} = \vec{u}$.

Elements of V are called **vectors**.

Examples of real vector spaces for the usual operations:

- The trivial vector space $\{0\}$,
- The set R^n of real $n-$tuples,
- The set $\mathbb{R}^{\mathbb{N}}$ of real sequences,
- The set $\mathbb{R}[X]$ of real polynomials,
- The set of real functions,
- The set $M_{m,n}(\mathbb{R})$ of matrices of size $m \times n$.

Exercise: Consider the set $V = (0, \infty)$ together with the two operations defined by $x \star y = xy$ and $k \bullet x = x^k$, for all $x, y \in V$ and $k \in \mathbb{R}$ Show that $(V, \star, \bullet)$ is a real vector space

Theorem

Let $(V, +, \cdot)$ be a real vector space. We have

1. *The zero vector $\vec{0}$ defined in axiom 4 is unique.*
2. *For each $\vec{u}$, the vector $-\vec{u}$, negative of u, defined in axiom 5 is unique.*
3. $\forall \vec{u} \in V$, *we have* $0 \cdot \vec{u} = \vec{0}$.
4. $\forall k \in \mathbb{R}$, *we have* $k \cdot \vec{0} = \vec{0}$.
5. $\forall \vec{u} \in V$, *we have* $(-1) \cdot \vec{u} = -\vec{u}$ *and* $-(-\vec{u}) = \vec{u}$.
6. $\forall k \in \mathbb{R}, \forall \vec{u} \in V$, *if* $k \cdot \vec{u} = \vec{0}$, *then* $k = 0$ *or* $\vec{u} = \vec{0}$.

Proofs:

1. Assume there exist $\vec{0}_1, \vec{0}_2$, both satisfying the axiom 4.
   For all $\vec{u} \in V$, we have
   $4a$: $\vec{u} + \vec{0}_1 = \vec{0}_1 + \vec{u} = \vec{u}$ and $4b$: $\vec{u} + \vec{0}_2 = \vec{0}_2 + \vec{u} = \vec{u}$.
   Take $\vec{u} = \vec{0}_2$ in $4a$ and $\vec{u} = \vec{0}_1$ in $4b$ to get

$$\vec{0}_1 \underset{4a}{=} \vec{0}_1 + \vec{0}_2 \underset{4b}{=} \vec{0}_2.$$

2. Let $\vec{u} \in V$ and assume there exist $\vec{v}_1, \vec{v}_2 \in V$, both satisfying axiom 5. That is
   $5a: \ \vec{u} + \vec{v}_1 = \vec{v}_1 + \vec{u} = \vec{0}$ and $5b: \ \vec{u} + \vec{v}_2 = \vec{v}_2 + \vec{u} = \vec{0}$.
   We have
   $\vec{v}_1 \underset{4}{=} \vec{v}_1 + \vec{0} \underset{5b}{=} \vec{v}_1 + (\vec{u} + \vec{v}_2) \underset{3}{=} (\vec{v}_1 + \vec{u}) + \vec{v}_2 \underset{5a}{=} \vec{0} + \vec{v}_2 \underset{4}{=} \vec{v}_2.$
3. Let $\vec{u} \in V$. We have

$$\begin{aligned} 0 \cdot \vec{u} &\underset{4}{=} 0 \cdot \vec{u} + \vec{0} \underset{5}{=} 0 \cdot \vec{u} + \left(0 \cdot \vec{u} + (-0 \cdot \vec{u})\right) \\ &\underset{3}{=} (0 \cdot \vec{u} + 0 \cdot \vec{u}) + (-0 \cdot \vec{u}) \underset{8}{=} (0 + 0) \cdot \vec{u} + (-0 \cdot \vec{u}) \\ &= 0 \cdot \vec{u} + (-0 \cdot \vec{u}) \underset{5}{=} \vec{0}. \end{aligned}$$

4. Let $k \in \mathbb{R}$. We have

$$\begin{aligned} k \cdot \vec{0} &\underset{4}{=} k \cdot \vec{0} + \vec{0} \underset{5}{=} k \cdot \vec{0} + \left(k \cdot \vec{0} + (-k \cdot \vec{0})\right) \\ &\underset{3}{=} (k \cdot \vec{0} + k \cdot \vec{0}) + (-k \cdot \vec{0}) \underset{7}{=} k \cdot (\vec{0} + \vec{0}) + (-k \cdot \vec{0}) \\ &\underset{4}{=} k \cdot \vec{0} + (-k \cdot \vec{0}) \underset{5}{=} \vec{0}. \end{aligned}$$

5. Let $\vec{u} \in V$. Using paragraph 3 of this theorem, we have

$$(-1) \cdot \vec{u} + \vec{u} \underset{10}{=} (-1) \cdot \vec{u} + 1 \cdot \vec{u} \underset{8}{=} (-1+1) \cdot \vec{u} = 0 \cdot \vec{u} = \vec{0},$$

and

$$\vec{u} + (-1) \cdot \vec{u} \underset{10}{=} 1 \cdot \vec{u} + (-1) \cdot \vec{u} \underset{8}{=} (1-1) \cdot \vec{u} = 0 \cdot \vec{u} = \vec{0}.$$

Hence $(-1) \cdot \vec{u}$ satisfies axiom 5 for $\vec{u}$. But we have already proved that this vector is unique. This gives $(-1) \cdot \vec{u} = -\vec{u}$. Using this, we obtain
$-(-\vec{u}) = (-1) \cdot ((-1) \cdot \vec{u}) \underset{9}{=} \big((-1) \times (-1)\big) \cdot \vec{u} = 1 \cdot \vec{u} \underset{10}{=} \vec{u}.$

6. Let $k \in \mathbb{R}$ and $\vec{u} \in V$ such that $k \cdot \vec{u} = \vec{0}$. Assume $k \neq 0$. Using paragraph 4 of this theorem, we have

$$\vec{u} \underset{10}{=} 1 \cdot \vec{u} = \left(\frac{1}{k} \times k\right) \cdot \vec{u} \underset{9}{=} \frac{1}{k} \cdot (k \cdot \vec{u}) = \frac{1}{k} \cdot \vec{0} = \vec{0}.$$

Definition (Subspaces)

Let $(V, +, \cdot)$ be a real vector space and $W \subseteq V$. We say that W is a **subspace** of V if $(W, +, \cdot)$ is a real vector space.

Some of the axioms in the definition of vector space are properties of the operations. They are automatiquely inherited to any subset of the space. These are axioms 2,3,7,8,9,10. Therefore, we only need to check that the subset satisfies the left axioms 1,4,5,6. Using the previous theorem, axioms 4 and 5 can be deduced from axiom 6, by assuming the subset nonempty and taking k in axiom 6 equals 0 and -1, respectively. We deduce the following theorem.

Theorem

Let $(V, +, \cdot)$ be a real vector space and $W \subseteq V$. The subset W is a subspace of V if and only if

1. *The subset W is not empty.*
2. *For all $\vec{u}, \vec{v} \in W$, we have $\vec{u} + \vec{v} \in W$.*
3. *For all $k \in \mathbb{R}$ and all $\vec{u} \in W$, we have $k \cdot \vec{u} \in W$.*

Example 1: Let $W = \{(2t+3s, s, t, 4t-s) | s, t \in \mathbb{R}\}$. Say if W under the usual operations is a vector space or not?
Solution: Because $\mathbb{R}^4$ is a vector space and $W \subseteq \mathbb{R}^4$, it is enough to check if W is a subspace of $\mathbb{R}^4$:

1. Take $s = t = 0$ to see that $(0, 0, 0, 0) \in W$.
2. Let $\vec{u}_1, \vec{u}_2 \in W$. There exist $s_1, s_2, t_1, t_2 \in \mathbb{R}$ such that $\vec{u}_1 = (2t_1 + 3s_1, s_1, t_1, 4t_1 - s_1)$ and $\vec{u}_2 = (2t_2 + 3s_2, s_2, t_2, 4t_2 - s_2)$. We have

$$\vec{u}_1 + \vec{u}_2 = (2t_1 + 3s_1 + 2t_2 + 3s_2, s_1 + s_2, t_1 + t_2, 4t_1 - s_1 + 4t_2 - s_2)$$
$$= (2(t_1+t_2)+3(s_1+s_2), (s_1+s_2), (t_1+t_2), 4(t_1+t_2)-(s_1+s_2)) \in W.$$

3. Let $\vec{u} \in W$ and $k \in \mathbb{R}$. There exist $s, t \in \mathbb{R}$ such that $\vec{u} = (2t + 3s, s, t, 4t - s)$. We have $k\vec{u} = (2kt + 3ks, ks, kt, 4kt - ks) \in W$

This shows that, W is a subspace of $\mathbb{R}^4$ and therefore, a vector space.

Example 2: Let $W = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 - x = 0\}$. Say if W under the usual operations is a vector space or not?
Solution: Because $\mathbb{R}^2$ is a vector space and $W \subseteq \mathbb{R}^2$, it is enough to check if W is a subspace of $\mathbb{R}^2$:
We have $(1, 0) \in W$ while $2(1, 0) = (2, 0) \notin W$. Hence W is not a vector space.

Example 3: Let $W = \{(2t + 1, t - 1, t) | t \in \mathbb{R}\}$. Say if W under the usual operations is a vector space or not?
Solution: Because $\mathbb{R}^3$ is a vector space and $W \subseteq \mathbb{R}^3$, it is enough to check if W is a subspace of $\mathbb{R}^3$:
Notice that the equation $(2t + 1, t - 1, t) = (0, 0, 0)$ has no solution, which means that $(0, 0, 0) \notin W$. Hence W is not a vector space.

Examples of Subspaces

Example 4: Let $W = \left\{ \begin{pmatrix} 2s & t \\ -s & s+t \end{pmatrix} | s, t \in \mathbb{R} \right\}$. Say if W under the usual operations is a vector space or not?

Solution: Because $M_2(\mathbb{R})$ is a vector space and $W \subseteq M_2(\mathbb{R})$, it is enough to check if W is a subspace of $M_2(\mathbb{R})$:

1. Take $s = t = 0$ to see that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$.
2. Let $\vec{u}_1, \vec{u}_2 \in W$. There exist $s_1, s_2, t_1, t_2 \in \mathbb{R}$ such that $\vec{u}_1 = \begin{pmatrix} 2s_1 & t_1 \\ -s_1 & s_1 + t_1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 2s_2 & t_2 \\ -s_2 & s_2 + t_2 \end{pmatrix}$. We have $\vec{u}_1 + \vec{u}_2 = \begin{pmatrix} 2(s_1 + s_2) & (t_1 + t_2) \\ -(s_1 + s_2) & (s_1 + s_2) + (t_1 + t_2) \end{pmatrix} \in W$.
3. Let $\vec{u} \in W$ and $k \in \mathbb{R}$. There exist $s, t \in \mathbb{R}$ such that $\vec{u} = \begin{pmatrix} 2s & t \\ -s & s+t \end{pmatrix}$. We have $k\vec{u} = \begin{pmatrix} 2ks & kt \\ -ks & ks + kt \end{pmatrix} \in W$

This shows that, W is a subspace of $M_2(\mathbb{R})$ and therefore, a vector space.

Example 5: Let $\mathcal{P}_n = \mathbb{R}_n[X] = \{P \in \mathbb{R}[X] | \deg P \leq n\}$, for $n \geq 0$, be the set of real polynomials of degree less or equal to n. Say if $\mathcal{P}_n$ under the usual operations is a vector space or not?

Solution: Because the set of polynomials $\mathcal{P} = \mathbb{R}[X]$ is a vector space and $\mathcal{P}_n \subseteq \mathcal{P}$, it is enough to check if $\mathcal{P}_n$ is a subspace of $\mathcal{P}$:

1. Because the zero polynomial has degree $\deg 0 = -\infty \leq n$, we have $0 \in \mathcal{P}_n$.
2. Let $P_1, P_2 \in \mathcal{P}_n$. Because $\deg(P_1 + P_2) \leq \max\{\deg P_1, \deg P_2\} \leq n$, we have $P_1 + P_2 \in \mathcal{P}_n$
3. Let $P \in \mathcal{P}_n$ and $k \in \mathbb{R}^*$. We have $\deg(kP) = \deg P \leq n$ and therefore, $kP \in \mathcal{P}_n$.

This shows that $\mathcal{P}_n$ is a subspace of $\mathcal{P}$ and therefore, a vector space.

Example 6: Let $W = \{P \in \mathbb{R}[X] | \deg P \leq n\}$, for $n \geq 0$, be the set of real polynomials of degree n. Say if W under the usual operations is a vector space or not?

Solution: The zero polynomial has degree $-\infty$ and therefore, $0 \notin W$. Hence, W is not a vector space.

Example 7: Let $W = \{P \in \mathcal{P}_3 | P(1) = 0\}$. Say if W under the usual operations is a vector space or not?

Solution: Because $\mathcal{P}_3$ is a vector space and $W \subseteq \mathcal{P}_3$, it is enough to check if W is a subspace of $\mathcal{P}_3$:

1. Because $0(1) = 0$, we have $0 \in W$.
2. Let $P_1, P_2 \in W$. Because $(P_1 + P_2)(1) = P_1(1) + P_2(1) = 0$, we have $P_1 + P_2 \in W$
3. Let $P \in W$ and $k \in \mathbb{R}$. We have $(kP)(1) = kP(1) = 0$ and therefore, $kP \in W$.

This shows that W is a subspace of $\mathcal{P}_3$ and therefore, a vector space.

Example 8: Recall that in the plane $\mathbb{R}^2$, a line is given by $\mathcal{L} = \{(x, y) \in \mathbb{R}^2 | ax + by = c\}$, where a, b, c are given real numbers and $(a, b) \neq (0, 0)$. For which values of a, b, c, the line $\mathcal{L}$ is a vector space?

Solution: Because $\mathbb{R}^2$ is a vector space and $\mathcal{L} \subseteq \mathbb{R}^2$, it is enough to check if $\mathcal{L}$ is a subspace of $\mathbb{R}^2$:

1. To get $\vec{0} = (0, 0) \in \mathcal{L}$, we must have $a \cdot 0 + b \cdot 0 = c$. This means that if $c \neq 0$, the set $\mathcal{L}$ is not a vector space.

We consider now the case when $c = 0$.

2. Let $(x_1, y_1); (x_2, y_2) \in \mathcal{L}$. Because $a(x_1 + x_2) + b(y_1 + y_2) = ax_1 + by_1 + ax_2 + by_2 = 0 + 0 = 0$, we deduce that $(x_1, y_1) + (x_2, y_2) \in \mathcal{L}$
3. Let $(x, y) \in \mathcal{L}$ and $k \in \mathbb{R}$. Because $a(kx) + b(ky) = k(ax + by) = k \cdot 0 = 0$, we deduce that $k(x, y) \in \mathcal{L}$.

This shows that $\mathcal{L}$ is a subspace of $\mathbb{R}^2$ and therefore, a vector space.

Examples of Subspaces

Example 9: Recall that in the space $\mathbb{R}^3$, a line is given by $\mathcal{L} = \{(x_0 + \alpha t, y_0 + \beta t, z_0 + \gamma t) | t \in \mathbb{R}\}$, where (x_0, y_0, z_0) is a point on the line and $< \alpha, \beta, \gamma >$ is a vector parallel to the line. The line $\mathcal{L}$ is a vector space if and only if the origin $(0, 0, 0)$ is on the line.
As an example, consider $\mathcal{L} = \{(1 + 2t, 2 + 4t, 3 + 6t) | t \in \mathbb{R}\}$. If we take $t = -\frac{1}{2}$, we will see that $(0, 0, 0) \in \mathcal{L}$. Hence, it is a vector space.

Example 10: Recall that in the space $\mathbb{R}^3$, a plane is given by $\Pi = \{(x, y, z) \in \mathbb{R}^3 | ax + by + cz = d\}$, where a, b, c, d are given real numbers and $(a, b, c) \neq (0, 0, 0)$. The plane Π is a vector space if and only if the origin $(0, 0, 0)$ is on the plane. Therefore, Π is a vector space only when $d = 0$.

Intersection of Subspaces

Theorem

Let W_1 and W_2 be two vector subspaces of V. The intersection $W_1 \cap W_2$ is a subspace of V.

Proof:

1. We have $\vec{0} \in W_1$ and $\vec{0} \in W_2$, as subspaces of V. We deduce that $\vec{0} \in W_1 \cap W_2$.
2. Let $\vec{u}_1, \vec{u}_2 \in W_1 \cap W_2$. We have $\vec{u}_1 + \vec{u}_2 \in W_1$ and $\vec{u}_1 + \vec{u}_2 \in W_2$, as subspaces of V. We deduce that $\vec{u}_1 + \vec{u}_2 \in W_1 \cap W_2$.
3. Let $\vec{u} \in W_1 \cap W_2$ and $k \in \mathbb{R}$. We have $k\vec{u} \in W_1$ and $k\vec{u} \in W_2$, as subspaces of V. We deduce that $k\vec{u} \in W_1 \cap W_2$.

This proves that $W_1 \cap W_2$ is a subspace of V.

Intersection of Subspaces

Example 1: Let $W = \{P \in \mathcal{P}_3 | P(1) = P(2) = 0\}$. Show that W is a vector space.
Solution: Let $W_1 = \{P \in \mathcal{P}_3 | P(1) = 0\}$ and $W_2 = \{P \in \mathcal{P}_3 | P(2) = 0\}$. The set $W = W_1 \cap W_2$ is a vector space as intersection of two vector spaces W_1, W_2, as seen previously.

Example 2: Consider
$\mathcal{L} = \{(x, y, z) \in \mathbb{R}^3 | ax + by + cz = 0 \text{ and } a'x + b'y + c'z = 0\}$.
This is the intersection of two planes
$\Pi_1 = \{(x, y, z) \in \mathbb{R}^3 | ax + by + cz = 0\}$ and
$\Pi_2 = \{(x, y, z) \in \mathbb{R}^3 | a'x + b'y + c'z = 0\}$ in the space which is either a line or a plane if the two planes are identical. This shows again that $\mathcal{L}$ is a vector space.

The null space and the column space of a matrix

Remark: We prove by induction that the intersection $W_1 \cap W_2 \cap \cdots \cap W_n$ of n vector subspaces of V is a subspace of V.

Definition

Let A be an $m \times n$ matrix.
The **null space** of A is the set Null $A = \{X \in M_{n,1}(\mathbb{R}) | AX = 0\}$.
The **column space** of A is the set Col $A = \{AX | X \in M_{n,1}(\mathbb{R})\}$.

Example: Let $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Show that Null A is a subspace of $M_{4,1}$.

Solution: Notice that Null A is the set of solutions of the homogeneous linear system $\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 & = 0 \\ 2x_1 + x_2 + 3x_3 + x_4 & = 0 \\ x_1 + x_2 + x_3 & = 0 \end{cases}$. It is a subspace as intersection of three subspaces of $M_{4,1}$.

Theorem

Let A be an $m \times n$ matrix.

1. *Null A is a vector subspace of $M_{n,1}(\mathbb{R})$.*
2. *Col A is a vector subspace of $M_{m,1}(\mathbb{R})$.*

Proof: 1- As seen in the previous example, Null A is a vector subspace of $M_{n,1}(\mathbb{R})$ as intersection of m vector subspaces of $M_{n,1}(\mathbb{R})$.

2- To prove that Col A is a vector subspace of $M_{m,1}(\mathbb{R})$, we have:

1. $0 = A0 \in \text{Col } A$.
2. Let $Y_1, Y_2 \in \text{Col } A$. There exist $X_1, X_2 \in M_{n,1}(\mathbb{R})$ such that $Y_1 = AX_1$ and $Y_2 = AX_2$. We have $Y_1 + Y_2 = AX_1 + AX_2 = A(X_1 + X_2) \in \text{Col } A$.
3. Let $Y \in \text{Col } A$ and $k \in \mathbb{R}$. There exist $X \in M_{n,1}(\mathbb{R})$ such that $Y = AX$. We have $kY = kAX = A(kX) \in \text{Col } A$.

Therefore, Col A is a vector subspace of $M_{m,1}(\mathbb{R})$.

Sum of subspaces

Remark: Notice that if W_1 and W_2 are two vector subspaces of V none of them is contained in the other, then $W_1 \cup W_2$ is not a vector subspace of V. As an example, $W_1 = \{(x, 0) | x \in \mathbb{R}\}$ and $W_2 = \{(0, y) | y \in \mathbb{R}\}$ are vector subspaces of $\mathbb{R}^2$, but $W_1 \cup W_2 = \{(x, y) | xy = 0\}$ is not a vector subspace of $\mathbb{R}^2$.

Definition

Let W_1 and W_2 be two vector subspaces of V. We call the sum of these two subspaces, the set
$W_1 + W_2 = \{\vec{u}_1 + \vec{u}_2 | \vec{u}_1 \in W_1, \vec{u}_2 \in W_2\}$.

Theorem

Let W_1, W_2 and W be vector subspaces of V.

1. *The sum $W_1 + W_2$ is also a subspace of V.*
2. *If $W_1 \cup W_2 \subseteq W$, then $W_1 + W_2 \subseteq W$.*

Proof:

1. To prove that $W_1 + W_2$ is a subspace of V, we have
    1. Because $\vec{0} \in W_1, W_2$, we have $\vec{0} = \vec{0} + \vec{0} \in W_1 + W_2$.
    2. Let $\vec{u}, \vec{v} \in W_1 + W_2$. There exist $\vec{u}_1, \vec{v}_1 \in W_1$ and $\vec{u}_2, \vec{v}_2 \in W_2$, such that $\vec{u} = \vec{u}_1 + \vec{u}_2$ and $\vec{v} = \vec{v}_1 + \vec{v}_2$. We have, $\vec{u} + \vec{v} = (\vec{u}_1 + \vec{v}_1) + (\vec{u}_2 + \vec{v}_2) \in W_1 + W_2$, since $\vec{u}_1 + \vec{v}_1 \in W_1$ and $\vec{u}_2 + \vec{v}_2 \in W_2$.
    3. Let $\vec{u} \in W_1 + W_2$ and $k \in \mathbb{R}$. There exist $\vec{u}_1 \in W_1$ and $\vec{u}_2 \in W_2$, such that $\vec{u} = \vec{u}_1 + \vec{u}_2$. We have $k\vec{u} = k\vec{u}_1 + k\vec{u}_2 \in W_1 + W_2$, since $k\vec{u}_1 \in W_1$ and $k\vec{u}_2 \in W_2$.
2. Assume that $W_1 \cup W_2 \subseteq W$ and let $\vec{u} \in W_1 + W_2$. There exist $\vec{u}_1 \in W_1 \subseteq W$ and $\vec{u}_2 \in W_2 \subseteq W$, such that $\vec{u} = \vec{u}_1 + \vec{u}_2$. But W is a vector subspace of V. Therefore, $\vec{u} = \vec{u}_1 + \vec{u}_2 \in W$. This proves that $W_1 + W_2 \subseteq W$.

Linear combinations and linear span of a set of vectors

Definition

Let V be a vector space and $\vec{u}_1, \vec{u}_2, \dots \vec{u}_n \in V$. A **linear combinaison** of the vectors $\vec{u}_1, \vec{u}_2, \dots \vec{u}_n$ is any vector of the form $x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n$, where $x_1, x_2, \dots, x_n \in \mathbb{R}$.
The **span** of $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$ is the set of all linear combinaisons of $\vec{u}_1, \vec{u}_2, \dots \vec{u}_n$. It is denoted $Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$.

We have
$Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\} = \{x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n | x_1, x_2, \dots, x_n \in \mathbb{R}\}$.

Examples:

- Consider in $\mathbb{R}^2$ the vectors $\vec{u}_1 = (1,2)$; $\vec{u}_2 = (2,1)$. The vector $\vec{u} = (3,4) = \frac{5}{3}\vec{u}_1 + \frac{2}{3}\vec{u}_2$ is a linear combinaison of $\vec{u}_1, \vec{u}_2$.
- The vector $\vec{u} = (7,2)$ is a linear combinaison of $\vec{u}_1, \vec{u}_2$. To see this, solve the equation $\vec{u} = x\vec{u}_1 + y\vec{u}_2$, which is equivalent to the system $\begin{cases} x + 2y &= 7 \\ 2x + y &= 2 \end{cases}$. We obtain $\vec{u} = -\vec{u}_1 + 4\vec{u}_2$.

Theorem

Let V be a vector space and $\vec{u}_1, \vec{u}_2, \dots \vec{u}_n \in V$. The set $Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$ is a vector subspace of V.

Proof:

1. We have $\vec{0} = 0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_n \in Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$.
2. If $\vec{u} = x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n \in Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$ and $\vec{v} = y_1\vec{u}_1 + y_2\vec{u}_2 + \cdots + y_n\vec{u}_n \in Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$, then $\vec{u} + \vec{v} = (x_1 + y_1)\vec{u}_1 + (x_2 + y_2)\vec{u}_2 + \cdots + (x_n + y_n)\vec{u}_n \in Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$.
3. If $\vec{u} = x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n \in Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$ and $\lambda \in \mathbb{R}$, then $\lambda\vec{u} = \lambda x_1\vec{u}_1 + \lambda x_2\vec{u}_2 + \cdots + \lambda x_n\vec{u}_n \in Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$.

This proves that $Span\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$ is a vector subspace of V.

Examples: $Span\{(2,1);(1,2)\} = \mathbb{R}^2$ and $Span\{(2,3);(1,1)\} = \mathbb{R}^2$

Linear combinations and linear span of a set of vectors

- Linear Combination, Subspace Generated/span
- Theorem: Equality of spans of two families

Linear dependence and linear independence of a set of vectors

- Linearly independent, $\{s_1, \ldots, s_r\} \subset \mathbb{R}^n$.

Basis and dimension of a vector space

Coordinates of a vector with respect to a basis

Change of basis

Rank and nullity of a matrix