Math 244 - Linear Algebra

Chapter 4: Vector Spaces

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Definition of a real vector space and examples

A **real vector space** $(V, +, \cdot)$ is a nonempty set V together with two operations $+$, called **addition**, and $\cdot$, called **multiplication by a scalar**, satisfying the following axioms:

1. $\forall \vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} \in V$;
2. $\forall \vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} = \vec{v} + \vec{u}$;
3. $\forall \vec{u}, \vec{v}, \vec{w} \in V$, we have $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$;
4. There exists an element $\vec{0}$ in V, called a **zero vector**, such that $\forall \vec{u} \in V$, we have $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$;
5. $\forall \vec{u} \in V$, there exists an element $-\vec{u} \in V$, called a **negative** of $\vec{u}$, such that $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$;
6. $\forall k \in \mathbb{R}$, $\forall \vec{u} \in V$, we have $k \cdot \vec{u} \in V$;
7. $\forall k \in \mathbb{R}$, $\forall \vec{u}, \vec{v} \in V$, we have $k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$;
8. $\forall k_1, k_2 \in \mathbb{R}$, $\forall \vec{u} \in V$, we have $(k_1 + k_2) \cdot \vec{u} = k_1 \cdot \vec{u} + k_2 \cdot \vec{u}$;
9. $\forall k_1, k_2 \in \mathbb{R}$, $\forall \vec{u} \in V$, we have $(k_1 k_2) \cdot \vec{u} = k_1 \cdot (k_2 \cdot \vec{u})$;
10. $\forall \vec{u} \in V$, we have $1 \cdot \vec{u} = \vec{u}$.

Definition of a real vector space and examples

- Elements of V are called **vectors**.
- Examples: The trivial vector space $\{0\}$, R^n, the set $\mathbb{R}^{\mathbb{N}}$ of real sequences, the set $\mathbb{R}[X]$ of real polynomials, The set of real functions, The set $M_{m,n}(\mathbb{R})$ of matrices of size $m \times n$.

Theorem

Let $(V, +, \cdot)$ be a real vector space. We have

1. *The zero vector $\vec{0}$ defined in axiom 4 is unique.*
2. *For each $\vec{u}$, the vector $-\vec{u}$, negative of u, defined in axiom 5 is unique.*
3. $\forall \vec{u} \in V$, *we have* $0 \cdot \vec{u} = \vec{0}$.
4. $\forall k \in \mathbb{R}$, *we have* $k \cdot \vec{0} = \vec{0}$.
5. $\forall \vec{u} \in V$, *we have* $(-1) \cdot \vec{u} = -\vec{u}$.
6. $\forall k \in \mathbb{R}, \forall \vec{u} \in V$, *if* $k \cdot \vec{u} = \vec{0}$, *then* $k = 0$ *or* $\vec{u} = \vec{0}$.

Subspaces

- Definition, Characterization, Examples
- Intersection of Subspaces, Solution set of $Ax = 0$.

- Linear Combination, Subspace Generated/span
- Theorem: Equality of spans of two families

Linear dependence and linear independence of a set of vectors

- Linearly independent, $\{s_1, \ldots, s_r\} \subset \mathbb{R}^n$.

Basis and dimension of a vector space

Coordinates of a vector with respect to a basis

Change of basis

Rank and nullity of a matrix