Math 203 - Differential and Integral Calculus

Chapter 3: Vector Fields

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Vector Fields

1. Definition of a vector field on $\mathbb{R}^2$ and on $\mathbb{R}^3$.
2. Examples: Velocity field (Air flows, Ocean currents), Force fields (Gravitational field, Electric field, Magnetic field).
3. Sketch some vectors of the following vector fields:

$$(x, y) \mapsto (x, y), \qquad (x, y) \mapsto (y, -x), \qquad (x, y, z) \mapsto (0, 0, z).$$

4. A Gradient field: $\vec{F}(x, y, z) = \vec{\nabla} f(x, y, z)$. The field $\vec{F}$ is said **conservative** and f is called a **scalar potential function** for $\vec{F}$.

Line Integrals

Instead of integring over an interval $[a, b]$, we integrate over a smooth curve: A smooth curve $\mathcal{C}$ is a trajectory of a vector valued function

$$x = x(t) \quad y = y(t) \quad (z = z(t) \text{ in space}),$$

or equivalently $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} =< x(t), y(t), z(t) >$, on an interval $[a, b]$, where $\vec{r}$ has a derivative and $\vec{r}'$ is continuous.
A line integral of $f(x, y, z)$ over a smooth curve $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} f(x, y, z)ds = \lim_{n} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i.$$

For any smooth parametrization $t \mapsto (x(t), y(t), z(t))$, $a \leq t \leq b$, we have

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

$$= \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2} |\Delta t_i|.$$

Line Integrals

Therefore,

$$\int_{\mathcal{C}} f(x,y,z)ds = \int_a^b f(x,y,z)\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt, \text{ where } a < b.$$

Remark: If two parametrizations define the same curve, the evaluation of the line integral will not depend on the choice of the parametrization.

Example 1: Use two different parametrizations of $\mathcal{C}$, the upper half of the unit circle, to evaluate the line integral $\int_{\mathcal{C}} x^2 y \; ds$.

Example 2: Evaluate the line integral $\int_{\mathcal{C}} y \sin z \; ds$, where $\mathcal{C}$ is the circular helix given by the parametric equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$.

Line Integrals of Vector Fields

Definition

Let $\mathcal{C}$ be a smooth curve joining a point A to a point B and $\vec{T}$ the tangent unit vector valued function along $\mathcal{C}$. Let $\vec{r}(t)$, $a \leq t \leq b$, be a smooth parametrization of $\mathcal{C}$ with $\vec{r}(a) = \vec{OA}$ and $\vec{r}(b) = \vec{OB}$. Let $\vec{F}$ be a continuous vector field defined on C. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot \vec{r}\,'(t)\, dt = \int_C \vec{F} \cdot d\vec{r}.$$

Remark: In physics, this is precisely the **work done** by the force field $\vec{F}$ along the oriented curve $\mathcal{C}$.

Example 1: Find the work done by the force field $\vec{F}(x, y) = -x^2 y\vec{i} + xy^2\vec{j}$ in a moving particle along the quarter-circle $\vec{r}(t) = \cos t\ \vec{i} + \sin t\ \vec{j}$, $0 \leq t \leq \frac{\pi}{2}$.

The Fundamental theorem for Line Integrals

Theorem

Let $\mathcal{C}$ be a smooth curve joining a point A to a point B and $\vec{r}(t)$, $a \leq t \leq b$ a corresponding parametrization. Let f be a differentiable function of several variables whose gradient vector field $\vec{\nabla} f$ is continuous on $\mathcal{C}$. We have

$$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = f(B) - f(A).$$

Proof: Notice, by the chain rule, that $\frac{d}{dt} f(\vec{r}(t)) = \vec{\nabla} f \cdot \vec{v}(t)$.

Example: Find the work done by the gravitational force field $\vec{F}(\vec{r}) = -\frac{GmM}{r^3}\vec{r}$ in moving a particle with mass m from the point $(3, 4, 0)$ to the point $(1, 2, 2)$ (Hint. Use the potential function $f(\vec{r}) = \frac{GmM}{r}$).

Remark: For a conservative vector field in a domain $\mathcal{D}$, its line integral over any smooth curve in $\mathcal{D}$ from a point A to a point B is **independent of path** and depends only on the points A and B.

The Curl

The **curl** of a $\mathcal{C}^1-$ vector field

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

is defined by

$$\vec{Curl}\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

We sometimes use the notation $\vec{Rot}\vec{F}$ for the curl.
For a $\mathcal{C}^2-$ scalar potential $f(x,y,z)$, we have $\vec{\nabla} \times \left(\vec{\nabla} f\right) = \vec{0}$.

Conservative Vector Field

Theorem

Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on a simply connected domaine $\mathcal{D}$. $\vec{F}$ is conservative on $\mathcal{D}$ if and only if $\vec{\nabla} \times \vec{F} = \vec{0}$ on $\mathcal{D}$.

Example: Show that

$$\vec{F}(x,y,z) = (2xy^3z + \cos y + z\cos x + 2x)\vec{i} + (3x^2y^2z - x\sin y + e^z - 3y^2)\vec{j}$$

$$+(x^2y^3 + ye^z + \sin x + 2z)\vec{k}$$

is conservative and find a scalar potential for $\vec{F}$.

Green' Theorem

Theorem (Green)

Let $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a $\mathcal{C}^1$ vector field defined on a connected simply connected domain $\mathcal{D}$ of $\mathbb{R}^2$ whose boundary $\partial\mathcal{D}$ is a smooth simple curve oriented counterclockwise. We have

$$\oint_{\partial\mathcal{D}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d\mathcal{A}.$$

The symbole $\oint$ denotes a line integral over a closed curve. The line integral $\oint_{\partial\mathcal{D}} \vec{F} \cdot d\vec{r}$ is called the **circulation** of the vector field $\vec{F}$ around the oriented closed curve $\partial\mathcal{D}$.

Example: Evaluate the integral $\oint_{\mathcal{C}} \left(2xy^3\vec{i} + (3x^2y^2 + x)\vec{j} \right) \cdot d\vec{r}$ in two ways (by direct computation and by using Green' theorem), where $\mathcal{C}$ is the unit circle oriented counter clockwise.

Oriented Surfaces

Definition (Oriented Surfaces)

In the space, at each point of a smooth surface, there are two unit normal vectors to the tangent plane $\pm\vec{n}$. A smooth surface in the space is said to be orientable (or two-sided surface) if we can choose at each point one of the two vectors in a continuous way.

Möbius Strip is an example of a non-orientable surface (one-sided surface).
If an orientable surface is given by the parametrization $\vec{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$, the two orientations are given by $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ and $\dfrac{\vec{r}_v \times \vec{r}_u}{\|\vec{r}_v \times \vec{r}_u\|}$, respectively.

Surface Integrals of Vector Fields

Definition

Let $\vec{F}$ be a continuous vector field defined on an oriented smooth closed and bounded surface $\mathcal{S}$. The surface integral of $\vec{F}$ over $\mathcal{S}$ is defined by

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = \int \int_{\mathcal{S}} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dudv,$$

where $\vec{r}$ is a parametrization of $\mathcal{S}$ and $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$.

This integral is also called the **flux** of $\vec{F}$ across $\mathcal{S}$.

Example: Evaluate the integral $\iint_{\mathcal{S}} \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{S}$, where $F(x, y, z) =< xy, yz, zx >$ and $\mathcal{S}$ is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies inside the quarter-space $x, z \geq 0$, oriented upward.

Stokes' Theorem

This is a generalization of Green' theorem to dimension 3.

Theorem (Stokes')

Let $F(x,y,z)$ be a $\mathcal{C}^1$ vector field defined on an oriented connected simply connected surface $\mathcal{S}$ of $\mathbb{R}^3$ whose boundary $\partial\mathcal{S}$ is a smooth simple curve whose orientation is induced from the orientation of $\mathcal{S}$. We have

$$\oint_{\partial\mathcal{S}} \vec{F}\cdot\vec{T}\ ds = \iint_{\mathcal{S}} \left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{n}\ d\mathcal{A}.$$

Example: Use Stokes' theorem to evaluate the integral $\iint_{\mathcal{S}} \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{S}$, where $F(x,y,z) =< xy, yz, zx >$ and $\mathcal{S}$ is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies inside the quarter-space $x, z \geq 0$, oriented upward.
This theorem says that the surface integral of $\vec{\nabla}\times\vec{F}$ depends only on the boundary of the surface and not on the surface itself.

The Divergence

The **divergence** of a $\mathcal{C}^1-$ vector field

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

is defined by

$$div\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

For a $\mathcal{C}^2-$ scalar potential $f(x,y,z)$, we have $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{F}\right) = 0$.

Theorem

Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on a simply connected domaine $\mathcal{D}$. There exists a $\mathcal{C}^1$ vector field $\vec{G}$ such that $\vec{F} = \vec{\nabla}\times\vec{G}$ on $\mathcal{D}$ if and only if $\vec{\nabla}\cdot\vec{F} = 0$ on $\mathcal{D}$.

$\vec{G}$ is called a **vector potential field** for $\vec{F}$.

The Divergence

Example: Evaluate the surface integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = xz\vec{i} - yz\vec{j} + x^2y^2\vec{k}$ and $\mathcal{S}$ is the surface defined on the square $-1 \leq x, y \leq 1$ by $z = \cos\frac{7\pi x}{2}\sin(4\pi y)$, oriented upward.

The existence of a Vector Potential Field with One Zero Component:

Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on a simply connected domaine $\mathcal{D}$ such that $\vec{\nabla} \cdot \vec{F} = 0$ on $\mathcal{D}$. How to find a vector potential field $\vec{G}$ for $\vec{F}$?

- Notice that if $\vec{F} = \vec{\nabla} \times \vec{G}_0 = \vec{\nabla} \times \vec{G}_1$, then $\vec{G}_1 = \vec{G}_0 + \vec{\nabla} f$, for some scalar potential f. Hence, if $\vec{G}_0$ is a particular solution of $\vec{F} = \vec{\nabla} \times \vec{G}$, the general solution has the form $\vec{G}_0 + \vec{\nabla} f$, for some scalar potential f.
- Given a continuous scalar field $r(x, y, z)$, the gradient of the scalar potential field $f(x, y, z) = -\int r(x, y, z)\, dz$ has its last component $-r(x, y, z)$. Hence, there exists a particular solution to $\vec{F} = \vec{\nabla} \times \vec{G}$ of the form $\vec{G}_0 = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j}$.

A method to Find a Vector Potential Field with One Zero Component:

- Let $\vec{F}(x,y,z) = p(x,y,z)\vec{i} + q(x,y,z)\vec{j} + r(x,y,z)\vec{k}$ satisfying $\vec{\nabla} \cdot \vec{F} = 0$ and $\vec{G}_0 = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j}$ be a vector potential field of $\vec{F}$.
- The equation $\vec{F} = \vec{\nabla} \times \vec{G}_0$ is equivalent to

$$-\frac{\partial Q}{\partial z} = p, \qquad \frac{\partial P}{\partial z} = q, \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = r.$$

- A particular solution is given by

$$P(x,y,z) = \int_a^z q(x,y,t)dt + \phi(x,y), \quad Q(x,y,z) = -\int_a^z p(x,y,t)dt.$$

- To choose $\phi(x,y)$, we use the third equation.

A method to find a vector potential field with one zero component:

Example 1: Show that $\vec{F}(x,y,z) = xz\vec{i} - yz\vec{j} + xy\vec{k}$ has a vector potential field and give an example of such a vector field.
Solution: We have $div\vec{F} = z - z + 0 = 0$ and $\vec{F}$ is defined on $\mathbb{R}^3$. Hence, $\vec{F}$ has a vector potential field $\vec{G}_0 = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j}$.
A solution of the partial equations is given by
$P(x,y,z) = -\frac{1}{2}yz^2 + \phi(x,y)$ and $Q(x,y,z) = -\frac{1}{2}xz^2$ and we have

$$xy = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\frac{1}{2}z^2 + \frac{1}{2}z^2 - \phi_y(x,y) = -\phi_y(x,y).$$

We chose $\phi(x,y) = -\frac{1}{2}xy^2$ and we have $\vec{G}_0 = -\frac{yz^2 + xy^2}{2}\vec{i} - \frac{xz^2}{2}\vec{j}$.
We check by computing $\vec{\nabla} \times \vec{G}_0$ and we find $\vec{F}$.

Back to Stokes' Theorem

Example: Use Stokes' Theorem to evaluate the surface integral $\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = xz\vec{i} - yz\vec{j} + xy\vec{k}$ and $\mathcal{S}$ is the surface defined on $x^2 + y^2 \leq 1$ by $z = xy$, oriented upward.

The Divergence Theorem

Theorem (Divergence')

Let $F(x, y, z)$ be a $\mathcal{C}^1$ vector field defined on an open region containing a simple solid bounded region $\mathcal{V}$ of $\mathbb{R}^3$ whose boundary $\partial\mathcal{V}$ is a smooth surface oriented outward. We have

$$\oiint_{\partial\mathcal{V}} \vec{F} \cdot \vec{n}\, dS = \iiint_{\mathcal{V}} \vec{\nabla} \cdot \vec{F}\, d\mathcal{V}.$$

This theorem is also called **Gauss'** theorem and **Ostrogradsky'** theorem.

Example 1: Evaluate, in two ways, directly and by using divergence' theorem, the integral $\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$, where $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ and $\mathcal{S}$ is the unit sphere, oriented outward.

Example 2: Evaluate the integral $\iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$, where $\vec{F}(x, y, z) = xy\vec{i} + (y^2 + e^{xz^2})\vec{j} + \sin xy\vec{k}$ and $\mathcal{S}$ is the surface of the region bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$.

Physical Interpretation of The Divergence and The Curl

From divergence' theorem:

- If $\vec{\nabla} \cdot \vec{F}$ is positive at a point, the flux at this point is positive. The point acts as a **source** for the field.
- If $\vec{\nabla} \cdot \vec{F}$ is negative at a point, the flux at this point is negative. The point acts as a **sink** for the field.
- If $\vec{\nabla} \cdot \vec{F}$ is zero at a point, there are no monopole at this point for the field.

From Stokes' theorem:

- If $\vec{\nabla} \times \vec{F}$ is nonzero at a point, the circulation at this point is nonzero. This measures the rotational motion of the field at this point, its direction and magnitude.
- If $\vec{\nabla} \times \vec{F}$ is zero at a point, the field is said irrotational at this point.

Maxwell's Equations in Electromagnetism

$\vec{E}$: Electric field,
ρ: Electric charge density,
ϵ_0: Permittivity of free space,

$\vec{B}$: Magnetic field,
$\vec{J}$: Current density,
μ_0: Permeability of free space.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \qquad \vec{\nabla} \cdot \vec{B} = 0,$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right).$$

- Electric charges produce electric fields. A positive charge acts as a source, and a negative charge acts as a sink.
- There are no magnetic monopoles. Magnetic field lines do not start or end; they always form continuous, closed loops.
- A changing magnetic field induces a curling electric field.
- Magnetic fields are generated either by moving electric charges (electric currents) or by changing electric fields.