Math 244 - Linear Algebra

Chapter 3: Linear Systems

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Definition of a linear System

Definition

- A **linear equation** is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_kx_k = b$, where $a_1, a_2, \ldots, a_k$, called coefficients, and b, called constant term, are given numbers, and $x_1, x_2, \ldots, x_k$ are unknowns.
- A **linear system** or a **system of linear equations**, is a collection of linear equations involving the same unknowns $x_1, x_2, \ldots, x_k$.
- A **solution** to a linear system is an ordered $k-$tuple $(s_1, s_2, \ldots, s_k)$ of given numbers such that the substitutions $x_1 = s_1, x_2 = s_2, \ldots, x_k = s_k$ make all the equations satisfied.
- **Solving** a linear system is finding all possible solutions to the system.

Definition of a linear System

Examples:

- $2x - 3y = 1$, $5x + \frac{1}{5}y + \pi z = 3$, $3x_1 + \sqrt{2}x_2 - e^5 x_3 + 5x_4 = 2$ are all linear equations.
- $\begin{cases} 2x + 3y &= 1 \\ x - y &= 2 \end{cases}$, $\begin{cases} x + 2y &= 2 \\ 2x - 5y &= 1 \\ 3x + 5y &= 7 \end{cases}$, $\begin{cases} x + 2y + z &= 1 \\ 2x + y &= 2 \\ x + y + z &= 1 \end{cases}$, $\begin{cases} x + y + z + w &= 1 \\ 2x - y + 3z + 2w &= 3 \end{cases}$ are linear systems.
- To the system $\begin{cases} 2x + 3y &= 1 \\ x - y &= 3 \end{cases}$, $(-1, 1)$ is not a solution, but $(2, -1)$ is a solution.

Matrix form of a linear System

Remark

Any linear system can be written in a matrix form $AX = B$, where A is the matrix of the coefficients, X the column matrix of the unknowns, and B the column matrix of the constant terms.

Examples:

- $\begin{cases} 2x + 3y &= 1 \\ x - y &= 2 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$
- $\begin{cases} x + y + z + w &= 1 \\ 2x - z + 2w &= 3 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

Solving a linear System

Method

To solve a linear system $AX = B$, we have two possibilities:

1. *Either the matrix A is invertible. In this case, the solution is given by $X = A^{-1}B$. To compute, $A^{-1}B$, we have two methods:*
   1. *Using elementary row operations on the matrix $[A|B]$ to obtain $\left[I|A^{-1}B\right]$ (Gauss-Jordan Elimination Method).*
   2. *Using $A^{-1}B = \frac{1}{\det A}Adj(A)B$ (Cramer's Rule).*
2. *Or the matrix A is not invertible. In this case, we use elementary row operations on the matrix $[A|B]$ to solve the system (Gauss/Gauss-Jordan Elimination Method).*

The matrix A is called the **matrix of the system** and the matrix $[A|B]$ the **augmented matrix of the system**.

Gauss and Gauss – Jordan Elimination Methods

To solve a linear system by using one of the two elimination methods:

Write the system in its augmented matrix form $[A|B]$ and then use one of the two methods:

- **Gauss Elimination Method**:
  1. Perform elementary row operations on $[A|B]$ to obtain a matrix in a **row echelon form**;
  2. Write the obtained matrix as a linear system and use back substitution method to find the solutions.
- **Gauss-Jordan Elimination Method**:
  1. Perform elementary row operations on $[A|B]$ to obtain a matrix in a **reduced row echelon form**;
  2. Read the solutions directly from the obtained matrix.

Gauss and Gauss – Jordan Elimination Methods

Example 1: Solve the linear system $\begin{cases} 3x+2y+z &= 2 \\ x+2y+3z &= 2 \\ x-y+2z &= 4 \end{cases}$.

Solution: We perform elementary row operations on the augmented matrix of the system:

$$\left(\begin{array}{rrr|r} 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & -1 & 2 & 4 \end{array}\right) \underset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 3 & 2 & 1 & 2 \\ 1 & -1 & 2 & 4 \end{array}\right)$$

$$\underset{\substack{R_2-3R_1 \\ R_3-R_1}}{\sim} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -4 & -8 & -4 \\ 0 & -3 & -1 & 2 \end{array}\right) \underset{-\frac{1}{4}R_2}{\sim} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -1 & 2 \end{array}\right)$$

Gauss and Gauss – Jordan Elimination Methods

$$\underset{R_3+3R_2}{\sim}\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 5 \end{array}\right) \underset{\frac{1}{5}R_3}{\sim}\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ This matrix has a row echelon form

- **Gauss Method:**

  The corresponding system is $\left\{\begin{array}{rl} x+2y+3z & =2 \\ y+2z & =1 \\ z & =1 \end{array}\right.$.

  We proceed by back substitution:
  From $z=1$, we have $y+2=1$. That is $y=-1$.
  From $y=-1, z=1$, we have $x-2+3=2$. That is $x=1$.
  Hence, the system has a unique solution $(x,y,z)=(1,-1,1)$.

Gauss and Gauss – Jordan Elimination Methods

- **Gauss-Jordan Method:** We continue from the obtained matrix in a row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \underset{\substack{R_1-3R_3 \\ R_2-2R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\underset{R_1-2R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$ This matrix has a reduced row echelon form

We deduce that the system has a unique solution that we can read directly from the matrix $(x, y, z) = (1, -1, 1)$.

Gauss and Gauss – Jordan Elimination Methods

Example 2: Solve the linear system $\begin{cases} 3x + 2y + z &= 2 \\ x + 2y + 3z &= 2 \\ x + y + z &= 2 \end{cases}$.

Solution: We perform elementary row operations on the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right) \underset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\underset{\substack{R_2 - 3R_1 \\ R_3 - R_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -4 & -8 & -4 \\ 0 & -1 & -2 & 0 \end{array}\right) \underset{-\frac{1}{4}R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

Gauss and Gauss – Jordan Elimination Methods

$$\underset{R_3+R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

This matrix has a row with leading 1 at the last column

This corresponds to the equation $0 = 1$ which has no solution. Therefore, the system has no solution. We can stop here and we don't need to go further.

Gauss and Gauss – Jordan Elimination Methods

Example 3: Solve the linear system $\begin{cases} 3x + 2y + z &= 2 \\ x + 2y + 3z &= 2 \\ x + y + z &= 1 \end{cases}$.

Solution: We perform elementary row operations on the augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right) \underset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

$$\underset{\substack{R_2 - 3R_1 \\ R_3 - R_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -4 & -8 & -4 \\ 0 & -1 & -2 & -1 \end{array}\right) \underset{-\frac{1}{4}R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & -1 \end{array}\right)$$

Gauss and Gauss – Jordan Elimination Methods

$$\underset{R_3+R_2}{\sim}\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ This matrix has a row echelon form

- **Gauss Method:**

The corresponding system is $\left\{\begin{aligned} x + 2y + 3z &= 2 \\ y + 2z &= 1 \\ 0 &= 0 \end{aligned}\right.$.

Because there is no constraint for z, we can give z any value, say $z = t$. Here t is called a **parameter**. We have:
From $z = t$, $y + 2t = 1$. That is $y = 1 - 2t$.
From $y = 1 - 2t$ and $z = t$, $x + 2 - 4t + 3t = 2$ and $x = t$.
Hence, the system has infinitely many solutions

$$(x, y, z) = (t, 1 - 2t, t), \ t \in \mathbb{R}.$$

Gauss and Gauss – Jordan Elimination Methods

- **Gauss-Jordan Method:** We continue from the obtained matrix in a row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \underset{R_1-2R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This matrix has a reduced row echelon form

Rather than getting back to the system, to read the solution directly on the matrix, we introduce a parameter $z = t$ in this matrix. This gives

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & t \end{array}\right) \underset{\substack{R_1+R_3 \\ R_2-2R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 1-2t \\ 0 & 0 & 1 & t \end{array}\right)$$

We deduce that the system has infinitely many solutions

$$(x, y, z) = (t, 1-2t, t), \;\; t \in \mathbb{R}.$$

Gauss and Gauss – Jordan Elimination Methods

Here are all the possibilities for a linear system $AX = B$:
Perform row elementary operations on the augmented matrix of a linear system $AX = B$ to obtain a matrix in a row echelon form.

1. If the matrix in a row echelon form has a leading 1 at the last column, the system $AX = B$ has **no solution**. We say that the system is **inconsistent**.
2. If the matrix in a row echelon form has no leading 1 at the last column, the system $AX = B$ has solutions. We say that the system is **consistent**. Moreover:
   - If the matrix has a leading 1 at all the other columns, the system $AX = B$ has **a unique solution**.
   - If some of the other columns have no leading 1, we need to add a parameter for each of the corresponding unknown to obtain **infinitely many solutions** to the system $AX = B$.

Homogeneous systems of linear equations

Definition

A linear system is said to be homogeneous if all its constant terms are zero. Its matrix form is $AX = 0$.

$(0, 0, \ldots, 0)$ is always a solution to a homogeneous linear system. This is called the **trivial solution**. We are left with only two possibilities:

1. Once we know that the homogeneous system has a unique solution, it is the trivial one. No need to go further for the computations.
2. If the system has infinitely many solutions, we perform the operations till the end to find all these solutions as we do in the general case.

Cramer's Rule

Let us first compute, for $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ and $B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, the product $\mathrm{Adj}(A)B$. We have

$$\begin{pmatrix} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix},$$

where $a' = \begin{vmatrix} a & a_2 & a_3 \\ b & b_2 & b_3 \\ c & c_2 & c_3 \end{vmatrix}$, $b' = \begin{vmatrix} a_1 & a & a_3 \\ b_1 & b & b_3 \\ c_1 & c & c_3 \end{vmatrix}$ and $c' = \begin{vmatrix} a_1 & a_2 & a \\ b_1 & b_2 & b \\ c_1 & c_2 & c \end{vmatrix}$.

Cramer's Rule

The solution to a linear system $AX = B$, when A is invertible, is given by $X = A^{-1}B = \frac{1}{\det A}\text{Adj}(A)B$. More precisely

Theorem (Cramer's Rule)

A linear system $AX = B$ is called a Cramer' System if the matrix A is square and $\det A \neq 0$. In this case, the system has a unique solution $(x_1, x_2, \dots, x_k)$, given by the formula $x_i = \frac{\det A_i}{\det A}$, where A_i is the matrix obtained from A by replacing the i^{th} column by B, for $i = 1, 2, \dots, k$.

Cramer's Rule

Example 1: Solve, by using Cramer's rule, the linear system

$$\begin{cases} 2x - 3y &= 1 \\ x + 2y &= 4 \end{cases} .$$

Solution: This linear system can be written

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} .$$

Because $\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7 \neq 0$, the system is a Cramer' sytem and we have $\det A_x = \begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} = 14$ and $\det A_y = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7$. Hence, the unique solution (x, y) to the system is given by

$$x = \frac{\det A_x}{\det A} = \frac{14}{7} = 2 \qquad \text{and} \qquad y = \frac{\det A_y}{\det A} = \frac{7}{7} = 1.$$

Cramer's Rule

Example 2: Solve, by using Cramer's rule, the linear system

$$\begin{cases} 3x + 2y + z &= 2 \\ x + 2y + 3z &= 2 \\ x - y + 2z &= 4 \end{cases} .$$

Solution: This linear system can be written

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} .$$

Because $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = 12 + 6 - 1 - 2 + 9 - 4 = 20 \neq 0$, the system is a Cramer' sytem and we have

Cramer's Rule

$$\det A_x = \begin{vmatrix} 2 & 2 & 1 \\ 2 & 2 & 3 \\ 4 & -1 & 2 \end{vmatrix} = 8 + 24 - 2 - 8 + 6 - 8 = 20,$$

$$\det A_y = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 2 \end{vmatrix} = 12 + 6 + 4 - 2 - 36 - 4 = -20,$$

$$\det A_z = \begin{vmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 24 + 4 - 2 - 4 + 6 - 8 = 20.$$

Hence, the unique solution (x, y, z) to the system is given by

$$x = \frac{\det A_x}{\det A} = 1, \quad y = \frac{\det A_y}{\det A} = -1, \quad z = \frac{\det A_z}{\det A} = 1.$$